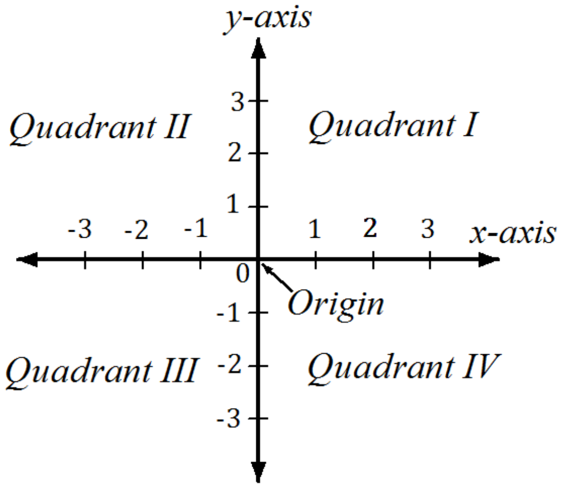
1.1 Cartesian coordinates

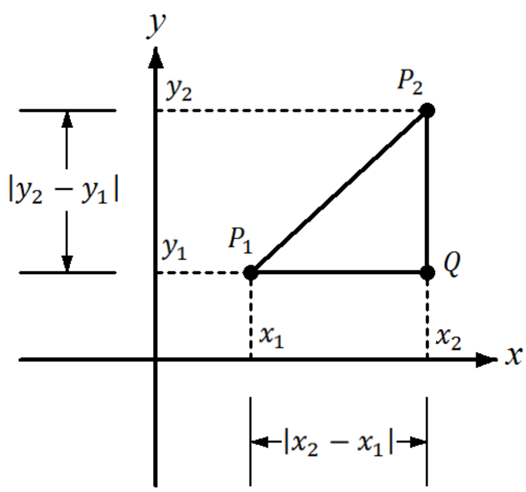
The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure. The horizontal real number line is usually called the x-axis, and the vertical real number line is usually called the y-axis. The point of intersection of these two axes is the origin, and the two axes divide the plane into four parts called quadrants.



Each point in the plane corresponds to an ordered pair (x, y) of real numbers and called the coordinates of the point.

1.2 Distance Formula

The distance formula is derived by using the Pythagorean Theorem. From the Figure:



$|P_1Q| = |x_2 - x_1|$, $|QP_2| = |y_2 - y_1|$

$|P_1P_2|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$

$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

The distance d between the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

1.3 The Midpoint Formula

The midpoint of the line segment joining the points in the coordinate plane is

$$\text{Midpoint}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) \qquad \text{where} \quad x=\frac{x_1+x_2}{2} \quad , y=\frac{y_1+y_2}{2}$$

Example1/Find (a) the distance between and (b) the midpoint the line segment joining, the points (-2,1) and (3,4).

Solution /

(a) $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(3-\left(-2\right)\right)^2+\left(4-1\right)^2}=\sqrt{25+9}=\sqrt{34}$

(b) $\text{Midpoint}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{-2+3}{2},\frac{1+4}{2}\right)=\left(\frac{1}{2},\frac{5}{2}\right)$

Example2/Use distance formula to show that the points A(1,1) , B(5,1) ,and C(5,7) are represent the vertices of right-angled triangle, then find the area of triangle.

Solution /

$AB=\sqrt{(5-1)^2+(1-1)^2}=\sqrt{(4)^2+(0)^2}=\sqrt{16}=4$

$BC=\sqrt{(5-5)^2+(7-1)^2}=\sqrt{(0)^2+(6)^2}=\sqrt{36}=6$

$CA=\sqrt{\left(5-1\right)^2+\left(7-1\right)^2}==\sqrt{(4)^2+(6)^2}=\sqrt{52}$

$(CA)^2=(AB)^2+(AB)^2 \Rightarrow \left(\sqrt{52}\right)^2=(4)^2+(6)^2 \Rightarrow 52=52$

$$A=\frac{1}{2}(\text{base})(\text{height})=\frac{1}{2}(4)(6)=12$$

1.4 The equation of line

An equation of the line passing through the point (x_1,y_1) and having sloped m.

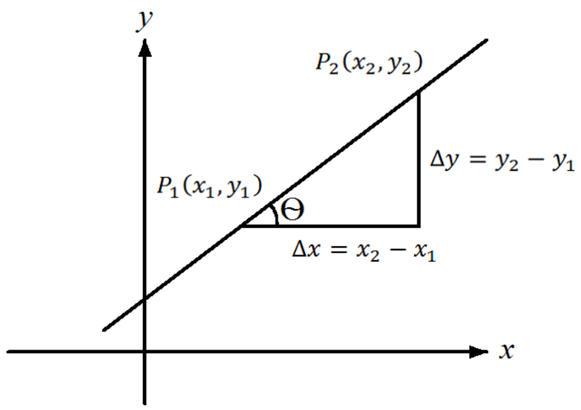
$$y-y_1=m(x-x_1)$$

Therefore we need a point (x_1,y_1) and the slope m.

1.5 The slope of line

$$m=\tan\theta$$

$$m=\frac{\Delta x}{\Delta y}=\frac{y_2-y_1}{x_2-x_1}$$



Notes:

❶ A horizontal line has slope zero because $\Delta y = 0$.

❷ The slope of a vertical line is undefined because $\Delta x = 0$.

Example1/ Find an equation of the line passing through the point (2,1) and having slope m=-1/2.

Solution /

$$y-y_1=m(x-x_1) \Rightarrow\Rightarrow \quad y-1=-\frac{1}{2}(x-2) \Rightarrow\Rightarrow \quad y=-\frac{1}{2}x+2$$

Example2/ Find an equation of the line passing through the points (-1,-2) and(2,3) .

Solution /

$$m=\frac{3-(-2)}{2-(-1)}=\frac{5}{3}$$

With $m=\frac{5}{3}$ and $(-1,-2)$

$$y-y_1=m(x-x_1)\Rightarrow\Rightarrow \quad y-(-2)=\frac{5}{3}(x-(-1))$$

$$\Rightarrow y+2=\frac{5}{3}x+\frac{5}{3} \quad \Rightarrow \quad \Rightarrow y=\frac{5}{3}x+\frac{5}{3}-2$$

1.6 Slope-Intercept Form of an Equation of a Line

A non vertical line crosses the y-axis at some point $(0,b)$. The number is called the y-intercept of the line

$$y-b=m(x-0)$$

$$\boxed{y=mx+b}$$

Which is called the slope-intercept form of an equation of a line.

Example1/ Find an equation of the line with slope $m=3/4$ and y-intercept 4.

Solution /

$$y=mx+b$$

$$\text{With } m=\frac{3}{4} \text{ and } b=4 \Rightarrow y=\frac{3}{4}x+4$$

1.7 The General Equation of a Line

An equation of the form

$$Ax+By+C=0$$

Where A,B , and C are constants and A and B are not both zero, is called a first-degree equation in x and y.

NOTE: We can find the slope by comparing the given equation with equation of intercept line $y=mx+b$.

$$\text{Or from } m=\frac{-A}{B}$$

Example1/ Find the slope of the line with equation $2x+3y+5=0$.

Solution /

$$3y=-2x-5 \Rightarrow\Rightarrow \quad y=\frac{-2}{3}x-\frac{5}{3} \quad \Leftrightarrow y=mx+b$$

The slope of the line $m=-\frac{2}{3}$ and y-intercept $b=-\frac{5}{3}$ OR $m=\frac{-A}{B}=\frac{-2}{3}$

1.8 Parallel Lines and Perpendicular Lines

Two lines L_1 and L_2 with slopes m_1 and m_2, respectively,

For parallel lines $\qquad m_1=m_2$

For perpendicular lines $\qquad m_1=-\frac{1}{m_2}$

Example1/ Find an equation of the line that passes through the point (6,7) and is perpendicular to the line with equation $2x+3y=12$.

Solution /

$$3y=-2x+12 \Rightarrow y=\frac{-2}{3}x+4 \Rightarrow \; y=mx+b$$

The slope of the line $m_2=-\frac{2}{3} \Rightarrow m_1=-\frac{1}{m_2}=-\frac{1}{-\frac{2}{3}}=\frac{3}{2}$

$$y-y_1=m(x-x_1) \Rightarrow\Rightarrow y-7=\frac{3}{2}(x-6) \Rightarrow\Rightarrow y-7=\frac{3}{2}x-9$$

$$\boxed{y=\frac{3}{2}x-2}$$

Homework

Q1/ Find the distance between given points:

$\boxed{1}$ (7,10), (1,2) $\boxed{2}$ (0,4), (−4,0) $\boxed{3}$ (t,4), (t,0) $\boxed{4}$ (−3,−5), (−7,−8)

$\boxed{5}$ $\left(\frac{-1}{2},\frac{-3}{2}\right)$, $\left(-3,\frac{-5}{2}\right)$ $\boxed{6}$ (a,b+1), (a+1,b)

Ans. $\boxed{1}$ 10 $\boxed{2}$ $4\sqrt{2}$ $\boxed{3}$ 4 $\boxed{4}$ 5 $\boxed{5}$ $\sqrt{29}/2$ $\boxed{6}$ $\sqrt{2}$

Q2/ Use distance formula to show that the given points are represent the vertices of right-angled triangle, then find the area of triangle.

$\boxed{1}$ $A(-1,-2)$, $B(3,-2)$, and $C(-1,-7)$ Ans. 10 sq.unit

$\boxed{2}$ $A(0,0)$, $B(-3,3)$, and $C(2,2)$ Ans. 6 sq.unit

$\boxed{3}$ $A(-2,-5)$, $B(9,1/2)$, and $C(4,21/2)$ Ans. 275/4 sq.unit

Q3/Show that the points $A(-2,-3)$, $B(3,-1)$, $C(1,4)$, and $D(-4,2)$ are the vertices of a square. Ans. $\sqrt{29}$

Q4/ If $A(-5,1)$, $B(-6,5)$, and $C(-2,4)$, determine whether triangle ABC is isosceles Ans. $AB=BC=\sqrt{17}$

Q5/Find the value of t so that the distance between the points $(-2,3)$, and (t,t) is 5 unit. Ans. $t=-2,3$

Q6/ If the point P (x,y) belongs to line passing through $P_1(-3,5)$ and $P_2(-1,2)$ satisfy $|PP_1|=4|P_1P_2|$. Fine the coordinate P .

Ans. $(5,-7)$, $(11,-16)$

Q7/If the distance between the points (3,y) and (8,7) is 13 find y.

Ans. $y=19,-5$

Q8/ Show that quadrilateral ABCD is a parallelogram, $A(-5,-2)$, $B(1,-1)$, $C(4,4)$ and $D(-2,3)$. Ans. : $m_{AB}=m_{CD}=1/6$, $m_{BC}=m_{AD}=5/3$

Q9/ Determine the equation of line passing through $(-4,3)$ and perpendiculars to $y=3x-5$ Ans. $x+3y-5=0$

Q10/ Find the value of k that make the slope of points $A(k,3)$,$B(-2,1)$ parallel to the slope of $c(5,-2)$, and $D(1,4)$. Ans. $k=-10/3$

Q11/Find the value of h that make the slope of points $A(h,3)$,$B(-2,1)$ perpendiculars the slope of $c(5,-2)$, and $D(1,4)$. Ans. $h=1$

Q12/Find the coordinates of a point equidistance from $(1,-6)$,$(5,-6)$ and $(6,-1)$. Ans. :$(3,-3)$

Q13/The line segment connecting $(x,6)$ and $(9,y)$ is bisected by the point $(7,3)$. Find the values of x and y. Ans. $x=5, y=0$

Q14/ If $(-2,-4)$ is the midpoint of $(6,-7)$ and (x,y). Find the values of x and y. Ans. $x=-10, y=-1$

Q15/What is the length of the line with a slope of 4/3 from a point $(6,4)$ to the y- axis. Ans. 10 unit

Q16/Find the equation of line passing through the origin and with a slope of 6 . Ans. $y-6x=0$

Q17/Determine B such that $3x+2y-7=0$ perpendicular to $2x-By+2=0$ Ans. $B=3$

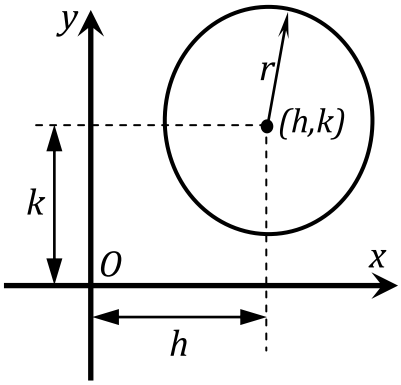
Q18/A line through $(-5,2)$ and $(1,-4)$ is perpendicular to the line through $(x,-7)$ and $(8,7)$. Find x . Ans. $x=-6$

Q19/Show that the three points A (2, 4), B (4, 6) and C (6, 8) are collinear. Ans. $m_{AB}=m_{BC}=m_{AC}=1$

Q20/ Find the area of triangle which the line $2x-3y+6=0$ forms with the coordinate axis Ans. 3 square unit

1.9 Circles

Circle is a locus of points that which moves so that it is equidistance from a fixed point called center.



✸Standard formula of circle

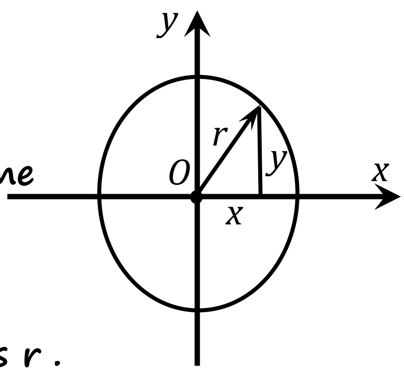
If $r>0$, and $C=(h,k)$ the equation of circle is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

Where r is the radius of the circle,

and h,k are the coordinates of its center.

Basic formula of circle



when $r>0$, and $C=(0,0)$ the equation of circle become

$$\boxed{x^2 + y^2 = r^2}$$

Example1/Determine the center C(h,k) and radius r .

$$(x-1)^2 + (y+1)^2 = 9$$

Solution /with comparing with standard formula $(x-h)^2 + (y-k)^2 = r^2$

$$h=1\ ,k=-1 \Rightarrow C=(1,-1) \quad , \quad r^2 = 9 \Rightarrow r=3$$

Example2/Determine the equation of circle with radius 3 and center C(-2,3) .

Solution / $(x-h)^2 + (y-k)^2 = r^2 \Rightarrow (x+2)^2 + (y-3)^2 = 9$

Example3/Determine the center C(h,k) and radius r of circle equation.

$$x^2 + y^2 + 2x + 8y - 8 = 0$$

Solution /

$$x^2 + 2x \qquad + y^2 + 8y \qquad = 8$$

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 + y^2 + 8y + \left(\frac{8}{2}\right)^2 = 8 + \left(\frac{2}{2}\right)^2 + \left(\frac{8}{2}\right)^2$$

$$x^2+2x+1+y^2+8y+16=8+1+16$$

$$(x+1)^2+(y+4)^2=25$$

With comparing with standard formula

$$h=-1\,,k=-4 \Rightarrow C=(-1,-4) \qquad , \quad r^2=25 \Rightarrow \Rightarrow=5$$

Example4/Determine the radius of a circle have center at $C(1,6)$ and containing the point $(-2,2)$.

Solution / $C(1,6)$ $h=1, k=6$, the point $(-2,2) \Rightarrow x=-2\,, y=2$ $r=?$

$$(x-h)^2+(y-k)^2=r^2$$

$$(-2-1)^2+(2-6)^2=r^2$$

$$(-3)^2+(-4)^2=r^2$$

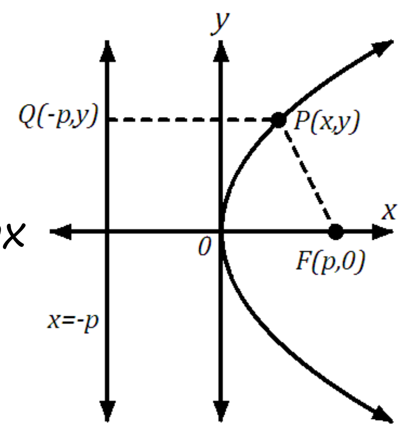
$$9+16=r^2 \Rightarrow\Rightarrow r^2=25 \Rightarrow\Rightarrow r=5$$

1.10 Parabolas

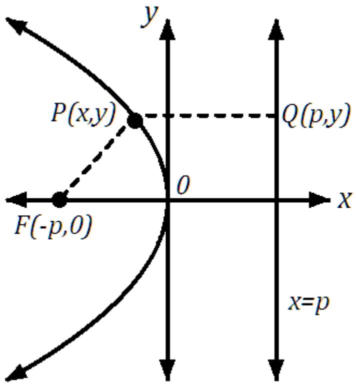
Parabola is a locus of a points that which moves so that it is always equidistance to a fixed point called focus and t a fixed straight line called directrix .

<u>Equations of parabola</u>

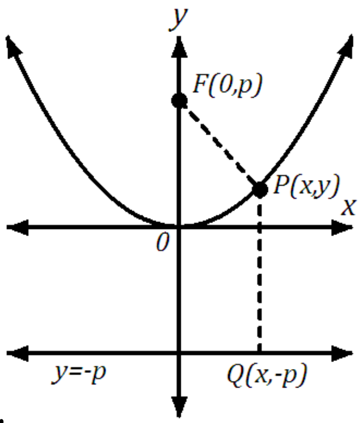
$y^2=4px$



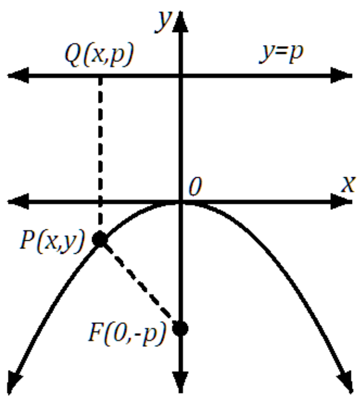
$y^2=-4px$



$x^2=4py$



$x^2=-4py$



Example1/ Determine the coordinates of focus and equation of directrix for the parabola $y^2=-8x$

Solution / with comparing with $y^2=-4px \Rightarrow 4p=8 \Rightarrow p=2$

$$F(-P,0)=(-2,0)$$

Directrix D: $x=2$

Example2/ Determine the coordinates of focus and equation of directrix for the parabola $x^2=-16y$

Solution / With comparing with $x^2=-4py \Rightarrow 4p=16 \Rightarrow p=4$

$$F(0,-P)=(0,-4)$$

Directrix D: $y=4$

1.11 Shifting Parabolas

The vertex is (h,k)

- x- axis

Open to the right: $(y-k)^2=4p(x-h)$

Open to the left: $(y-k)^2=-4p(x-h)$

- y- axis

Open upward: $(x-h)^2=4p(y-k)$

Open downward: $(x-h)^2=-4p(y-k)$

Example1/ Determine the coordinates of vertex, focus, and equation of directrix for the parabola $(y+1)^2=-12(x-2)$

Solution / with comparing with $(y-k)^2=-4p(x-h)$

$$h=2 \quad ,k=-1 \quad \Rightarrow \quad \text{vertice } V(h,k)=(2,-1)$$

$$4p=12 \quad \Rightarrow \quad p=3 \quad \Rightarrow \quad \text{focus } F(-p+h,k)=(-3+2,1)=(-1,1)$$

$$\text{Directrix} \quad D: \ x=p+h=3+2 \Rightarrow x=5$$

Example2/ Determine the coordinates of vertex, focus, and equation of directrix for the parabola $x^2+4x-10y+34=0$

Solution /

$$x^2+4x=10y-34 \quad \Rightarrow \quad x^2+4x+\left(\frac{4}{2}\right)^2=10y-34+\left(\frac{4}{2}\right)^2$$

$$x^2+4x+4=10y-30 \Rightarrow \quad (x+2)^2=10(y-3)$$

With comparing with $(x-h)^2=4p(y-k)$

$$h=-2 \quad ,k=3 \quad \Rightarrow \quad \text{vertxe } V(h,k)=(-2,3)$$

$$4p=10 \quad \Rightarrow \quad p=\frac{5}{2} \quad , \quad \Rightarrow \quad \text{focus } F(h,p+k)=\left(-2,\frac{5}{2}+3\right)=\left(-2,\frac{11}{2}\right)$$

$$\text{Directrix} \quad D: \ y=-p+k=-\frac{5}{2}+3 \Rightarrow y=\frac{1}{2}$$

Example3/ Determine the coordinates of vertex for the parabola

$$3y=4x^2+4x+5$$

Solution / $3y=4x^2+4x+5 \Rightarrow \quad 4x^2+4x=3y-5 \qquad \div 4$

$$x^2+x=\frac{3}{4}y-\frac{5}{4} \Rightarrow x^2+x+\left(\frac{1}{2}\right)^2=\frac{3}{4}y-\frac{5}{4}+\left(\frac{1}{2}\right)^2$$

$$x^2+x+\frac{1}{4}=\frac{3}{4}y-\frac{5}{4}+\frac{1}{4} \quad \Rightarrow \quad \left(x+\frac{1}{2}\right)^2=\frac{3}{4}y-1$$

$$\left(x+\frac{1}{2}\right)^2=\frac{3}{4}\left(y-\frac{4}{3}\right)$$

With comparing with $(x-h)^2=4p(y-k)$

$$h=-\frac{1}{2} \quad ,k=\frac{4}{3} \quad \Rightarrow \quad \text{vertxe } V(h,k)=\left(-\frac{1}{2},\frac{4}{3}\right)$$

Homework.

Q1/ Find the equation in standard form for the circles that satisfy the given conditions.

1. Radius 3 and center $(0,2)$ **Ans.** $(x-0)^2 + (y-2)^2 = 9$
2. Radius 2 and center $(-1,4)$ **Ans.** $(x+1)^2 + (y-4)^2 = 4$
3. Radius 5 and center $(3,4)$ **Ans.** $(x-3)^2 + (y-4)^2 = 25$

Q2/ Find the equation of circles with Radius 4 and containing the points $(-3,0)$ and $(5,0)$ **Ans.** $(x-1)^2 + (y-0)^2 = 16$

Q3/ Find the equation of circles where the points $(3,7)$ and $(-3,-1)$ are the end points of a diameter. **Ans.** $(x-0)^2 + (y-3)^2 = 25$

Q4/ Find the radius r and the coordinates (h,k) of the center of the circle for each equation

1. $(x+1)^2 + (y-2)^2 = 9$ **Ans.** $r=3$, $(h,k)=(-1,2)$
2. $(x+3)^2 + (y-10)^2 = 100$ **Ans.** $r=10$, $(h,k)=(-3,10)$
3. $x^2 + y^2 + 2x + 4y + 4 = 0$ **Ans.** $r=1$, $(h,k)=(-1,-2)$
4. $x^2 + y^2 - x - y - 1 = 0$ **Ans.** $r=\sqrt{3/2}$, $(h,k)=(1/2,1/2)$
5. $4x^2 + 4y^2 + 8x - 4y + 1 = 0$ **Ans.** $r=1$, $(h,k)=(-1,1/2)$
6. $3x^2 + 3y^2 - 6x + 9y = 27$ **Ans.** $r=7/2$, $(h,k)=(1,-3/2)$
7. $4x^2 + 4y^2 + 4x - 4y + 1 = 0$ **Ans.** $r=1/2$, $(h,k)=(-1/2,1/2)$

Q5/ Find the radius of the circle $x^2+y^2-6y=0$

Ans. $r=3$

Q6/ Find the diameter of the circle $9x^2+9y^2=16$

Ans. $r=8/3$

Q7/ How far from the y-axis to center of the curve $2x^2+2y^2+10x-6y-55=0$

Ans. $h=-2.5$

Q8/ What is the distance between the centers of circles $x^2+y^2+2x+4y-3=0$ and $x^2+y^2-8x-6y+7=0$ Ans. 7.07

Q9/The center of circle is at $(1,1)$ and one point on its circumference is $(-1,-3)$ find the other end of diameter through $(-1,-3)$ Ans. $(3,5)$

Q10/ Find the area of the circle whose equation is $x^2+y^2=6x-8y$ .Ans. 25π

Q11/Find the vertex of the parabola $x^2=4(y-2)$. Ans. $(0,2)$

Q12/Find the equation of the directrix of the parabola $y^2=16x$. Ans. $x=-4$

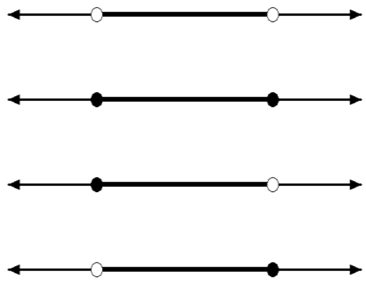
Q13/Find the vertix of the parabola $3x+2y^2-4y+7=0$. Ans. $(-5/3,1)$

Q14/Find the focus of the parabola $y^2+4x-4y-8=0$. Ans. $(2,2)$

1.12 Intervals

There are two types of intervals

$\boxed{A}$ Finite intervals



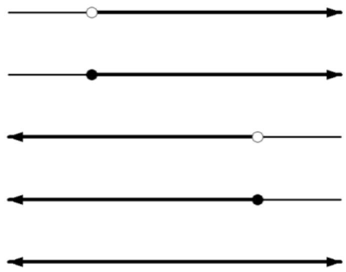
1) $(a, b) \Rightarrow a < x < b \Rightarrow open$

2) $[a, b] \Rightarrow a \le x \le b \Rightarrow closed$

3) $[a, b) \Rightarrow a \le x < b \Rightarrow half - open$

4) $(a, b] \Rightarrow a < x \le b \Rightarrow half - open$

$\boxed{B}$ Infinite intervals



1) $(a, \infty) \Rightarrow x > a \Rightarrow open$

2) $[a, \infty) \Rightarrow x \ge a \Rightarrow closed$

3) $(-\infty, b) \Rightarrow x < b \Rightarrow open$

4)$(-\infty, b] \Rightarrow x \le b \Rightarrow closed$

5)$(-\infty, \infty) \Rightarrow R \Rightarrow open\ and\ closed$

1.13 Inequalities

An inequality is any expression involving one of the symbols $<, >, \le or \ge$.

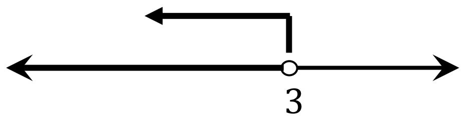
Example 1: Solve the following inequalities:

$(a)\ 10x < 18 + 4x$

$10x - 4x < 18$

$6x < 18$

$x < 3$



The solution set is open interval $(-\infty, 3)$

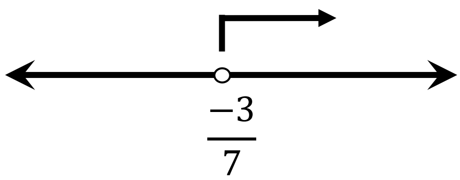
$(b) - \dfrac{x}{3} < 2x + 1 \qquad \times 3$

$-x < 6x + 3$

$0 < 7x + 3$

$-3 < 7x$

$\dfrac{-3}{7} < x$



The solution set is open interval $(\dfrac{-3}{7}, \infty)$

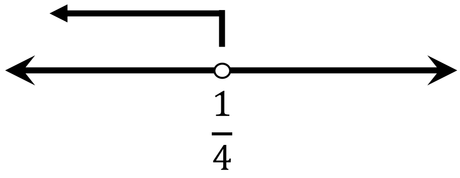
$(c)\ \dfrac{2}{x} - 4 < \dfrac{3}{x} - 8 \qquad\qquad \times x$

$2 - 4x < 3 - 8x$

$8x - 4x < 3 - 2$

$4x < 1$

$x < \dfrac{1}{4}$



The solution set is open interval $(-\infty, \frac{1}{4})$

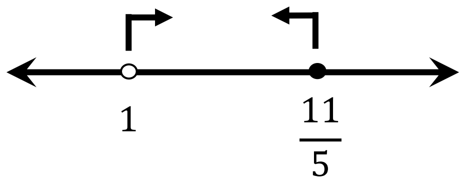
$(d)\ \dfrac{6}{x-1} \geq 5$

First $x - 1$ must be greater than zero

$x - 1 > 0 \Longrightarrow x > 1$

$\dfrac{6}{x-1} \geq 5 \Longrightarrow\ 6 \geq 5x - 5$

$11 \geq 5x \Longrightarrow \dfrac{11}{5} \geq x$



The solution set is half − open interval $(1, \frac{11}{5}]$

$(e)\ 2 \leq\ 5 - 3x < 11$

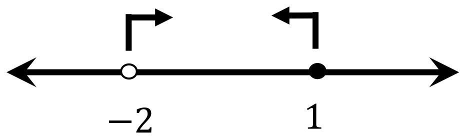
$2 - 5 \leq\ 5 - 5 - 3x < 11 - 5$

$-3 \leq\ -3x < 6$

$\dfrac{-3}{3} \leq \dfrac{-3x}{3} < \dfrac{6}{3} \qquad\qquad \div 3$

$-1 \leq\ -x < 2 \qquad\qquad \times -1$

$1 \geq\ x > -2$



The solution set is half − open interval $(-2,1]$

1.14 Absolute Value

The absolute value of a number x, denoted by $|x|$is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 1 : Find Absolute Values

(1) $|3| = 3$ (2) $|0| = 0$ (3) $|-5| = 5$

Absolute Value Properties

(1) $|-\mathrm{a}| = \mathrm{a}$

(2) $|\mathrm{a}b| = |\mathrm{a}||b|$

(3) $\left|\frac{\mathrm{a}}{b}\right| = \frac{|\mathrm{a}|}{|b|}$

(4) $|\mathrm{a} + b| < |a| + |b|$

Absolute Values and Intervals

If a is any positive number, then

(1) $|x| = \mathrm{a}$ if and only if $x = \pm\mathrm{a}$

(2) $|x| < \mathrm{a}$ if and only if $-\mathrm{a} < x < \mathrm{a}$

(3) $|x| > \mathrm{a}$ if and only if $x > \mathrm{a}$ *or* $x < -\mathrm{a}$

(4) $|x| \leq \mathrm{a}$ if and only if $-\mathrm{a} \leq x \leq \mathrm{a}$

(5) $|x| \geq \mathrm{a}$ if and only if $x \geq \mathrm{a}$ *or* $x \leq -\mathrm{a}$

Example 2 : Solve the following inequalities

(a) $|x - 3| = 2$

$x - 3 = 2$, $x - 3 = -2$

$x = 2 + 3$, $x = -2 + 3$

$x = 5$, $x = 1$

The solution set is $\{1,5\}$

$(b)\ |x-5| = 3x-1$

$x-5 = 3x-1 \quad , \quad x-5 = -(3x-1)$

$x-5 = 3x-1 \quad , \quad x-5 = -3x+1$

$-5+1 = 3x-x \quad , \quad -5-1 = -3x-x$

$-4 = 4x \quad , \quad -6 = -4x$

$-2 = x \quad , \quad \frac{3}{2} = x$

The solution set is $\left\{-2, \frac{3}{2}\right\}$

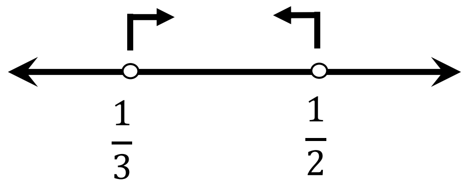
$(c)\ \left|5-\frac{2}{x}\right| < 1$

$-1 < 5-\frac{2}{x} < 1$

$-6 < -\frac{2}{x} < -4$

$3 < \frac{1}{x} < 2$

$\frac{1}{3} < x < \frac{1}{2}$



The solution set is open interval $(\frac{1}{3}, \frac{1}{2})$

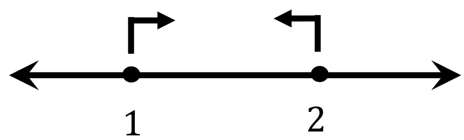
$(d)|2x-3| \leq 1$

$-1 \leq 2x-3 \leq 1$

$2 \leq 2x \leq 4$

$1 \leq x \leq 2$

The solution set is closed interval $[1,2]$



$(e)|2x-3| \geq 1$

$2x-3 \geq 1 \quad or \quad 2x-3 \leq -1$

$2x \geq 4 \quad or \quad 2x \leq 2$

$x \geq 2 \quad or \quad x \leq 1$

The solution set is $(-\infty, 1] \cup [2, \infty)$

Homework.

q❶/ Solve the following inequalities:

❶ $\frac{9}{4} < \frac{5}{2} + \frac{3}{2}x$ *Ans.* $(-6, \infty)$

❷ $3 < 5x \leq 2x + 11$ *Ans.* $(\frac{3}{5}, \frac{11}{3}]$

❸ $4 \leq 3x - 2 < 13$ *Ans.* $[2,5)$

❹ $3 > -4 - 4x \geq -8$ *Ans.* $(-\frac{7}{4}, 1]$

❺ $3\left(x + \frac{2}{3}\right) < 5(2x + 5)$ *Ans.* $(-\frac{23}{7}, \infty)$

❻ $\frac{3}{1-x} \leq 1$ *Ans.* $(-\infty, -2] \cup (1, \infty)$

❼ $\frac{3}{x-5} \leq 2$ *Ans.* $(-\infty, 5) \cup [\frac{13}{2}, \infty)$

❽ $-1 < \frac{5-3x}{4} \leq \frac{17}{2}$ *Ans.* $[-\frac{29}{3}, 3)$

❾ $2x - 5 < \frac{1}{3} + \frac{3}{4}x + \frac{1-x}{3}$ *Ans.* $(-\infty, \frac{68}{19})$

❿ $\frac{5}{x} = \frac{3}{4}$ *Ans.* $(-\infty, 0) \cup (\frac{20}{3}, \infty)$

⓫ $-5 \geq 6(x - 4) + 7$ *Ans.* $(-\infty, 2]$

⓬ $\frac{1}{2}(x - 6) - \frac{4}{3}(x + 2) \geq -\frac{3}{4}x - 2$ *Ans.* $(-\infty, -44]$

q❷/ Solve the following inequalities:

❶ $|3x+2|=5$ — *Ans.* $\left\{-\frac{7}{3},1\right\}$

❷ $|5x|=3-x$ — *Ans.* $\left\{-\frac{3}{4},\frac{1}{2}\right\}$

❸ $|2x-5|<1$ — *Ans.* $(2,3)$

❹ $|4x-6|\le 3$ — *Ans.* $\left[\frac{3}{4},\frac{9}{4}\right]$

❺ $|2x-5|>3$ — *Ans.* $(-\infty,1)\cup(4,\infty)$

❻ $|3-5x|\ge 5$ — *Ans.* $\left[-\frac{2}{5},\frac{8}{5}\right]$

❼ $|x+3|>0.5$ — *Ans.* $(-3.5,-2.5)$

❽ $\left|\frac{x+2}{x-2}\right|=5$ — *Ans.* $\left\{\frac{4}{3},3\right\}$

❾ $\left|\frac{3x+8}{2x-3}\right|=4$ — *Ans.* $\left\{\frac{4}{11},4\right\}$

❿ $|3x+2|=5-x$ — *Ans.* $\left\{\frac{-7}{2},\frac{3}{4}\right\}$

⓫ $|9x|-11=x$ — *Ans.* $\left\{\frac{-11}{10},\frac{11}{8}\right\}$

⓬ $2x-7=|x|+1$ — *Ans.* $\{8\}$

⓭ $|5-6x|\ge 9$ — *Ans.* $(-\infty,\frac{-2}{3})\cup(\frac{7}{3},\infty)$

⓮ $1<|x+2|<4$ — *Ans.* $(-6,-3)\cup(-1,2)$

⓯ $|x-1|+|x+1|\ge 4$ — *Ans.* $(-\infty,\frac{-5}{2})\cup(\frac{3}{2},\infty)$

2.1 Limits

Computing a limit just means computing what happens to the values of the function $f(x)$ if $f(x)$ is evaluated for values of x getting closer and closer to (but does not equal) the number c , if these values of $f(x)$ get closer and closer to one particular number L . You say that

The limit of $f(x)$, as x approaches c, equals L .

$$\lim_{x\to c} f(x) = L$$

2.1.1 Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

1 − $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$

2 − $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$

3 − $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) = L$ (Where c is any constant)

4 − $\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\cdot\left[\lim_{x\to a} g(x)\right] = L\cdot M$

5 − $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$, If $\lim_{x\to a} g(x) \neq 0$

6 − $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$

7 − $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$

8 − $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |L|$

9 − $\lim_{x\to a} c = c$ (Where a is any constant)

10 − $\lim_{x\to a} x = a$

Example 1 If $\lim_{x\to2} f(x) = 6$ and If $\lim_{x\to2} g(x) = 4$, find: –

① $\lim_{x\to2}[f(x)+g(x)]$ ② $\lim_{x\to2}[3f(x)-2g(x)]$

③ $\lim_{x\to2}\sqrt{f(x)\cdot g(x)}$ ④ $\lim_{x\to2}\left|\frac{f(x)}{g(x)}\right|$

Solution

① $\lim_{x\to2}[f(x)+g(x)] = \lim_{x\to2} f(x) + \lim_{x\to2} g(x) = 6+4 = 10$

② $\lim_{x\to2}[3f(x)-2g(x)] = \lim_{x\to2} 3f(x) - \lim_{x\to2} 2g(x) = 3\lim_{x\to2} f(x) - 2\lim_{x\to2} g(x)$

$$= 3(6) - 2(4) = 10$$

③ $\lim_{x\to2}\sqrt{f(x)\cdot g(x)} = \sqrt{\lim_{x\to2}[f(x)\cdot g(x)]} = \sqrt{\left[\lim_{x\to2} f(x)\right]\cdot\left[\lim_{x\to2} g(x)\right]} = \sqrt{6(4)} = \sqrt{24}$

④ $\lim_{x\to2}\left|\frac{f(x)}{g(x)}\right| = \left|\lim_{x\to2}\frac{f(x)}{g(x)}\right| = \left|\frac{\lim_{x\to2} f(x)}{\lim_{x\to2} g(x)}\right| = \left|\frac{6}{4}\right| = \left|\frac{3}{2}\right| = \frac{3}{2}$

Example 2: If $\lim_{x\to3} f(x) = 4$ and $\lim_{x\to3} g(x) = 8$, evaluate $\lim_{x\to3}\left[f^2(x)\cdot\sqrt[3]{g(x)}\right]$

Solution

$$\lim_{x\to3}\left[f^2(x)\cdot\sqrt[3]{g(x)}\right] = \left[\lim_{x\to3} f^2(x)\right].\left[\lim_{x\to3}\sqrt[3]{g(x)}\right] = \left[\lim_{x\to3} f(x)\right]^2.\left[\sqrt[3]{\lim_{x\to3} g(x)}\right]$$

$$= [4]^2.\left[\sqrt[3]{8}\right] = 32$$

2.1.2 Techniques for Evaluating Limits

1 – Direct substitution method,

2 – Factoring method,

3 – The conjugate method,

Usually, only one of these techniques will work on a given limit problem, so you should try one method at a time until you find one that works.

1. Direct substitution method

Limits can be evaluated simply by plugging the x value you're approaching into the function.

$$\boxed{\lim_{x\to a} c = c} \qquad \boxed{\lim_{x\to a} x = a}$$

Example 1 : $\lim_{x\to 3} 7 = 7$

Example 2 : $\lim_{x\to 3} x = 3$

Example 3 : $\lim_{x\to -5} x = -5$

Example 4 : Compute $\lim_{x\to 1} (x^2 + x + 2)$

Solution : $\lim_{x\to 1} (x^2 + x + 2) = (1)^2 + 1 + 2 = 4$

Example 5 : Compute $\lim_{x\to 2} \dfrac{x-2}{\sqrt{2+x}}$

Solution : $\lim_{x\to 2} \dfrac{x-2}{\sqrt{2+x}} = \dfrac{2-2}{\sqrt{2+2}} = \dfrac{0}{2} = 0$

Example 6 : Compute $\lim_{y\to 3} \sqrt[3]{\dfrac{y^2+5y+3}{y^2-1}}$

Solution : $\lim_{y\to 3} \sqrt[3]{\dfrac{y^2+5y+3}{y^2-1}} = \sqrt[3]{\dfrac{(3)^2+5(3)+3}{(3)^2-1}} = \sqrt[3]{\dfrac{27}{8}} = \dfrac{3}{2}$

2. Factoring Method

Some limits when we solving it by direct substitution the product yields to (0/0) , that's not allowed, because we can't have 0 in the denominator of a fraction, therefore we need another way to find the limit.

The best alternative to substitution is the factoring method. In this method we must simplified before plugging x value. The simplified will be by using one of the following:

1- $a^2 - b^2 = (a-b)(a+b)$

2- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

3- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

4- $ax^2 - bx = x(ax-b)$

5- $a^2 - 2ab + b^2 = (a-b)(a-b)$

6- $a^2 + 2ab + b^2 = (a+b)(a+b)$

Example 1 : Compute $\lim\limits_{x\to -1} \dfrac{x^2-1}{x+1}$

Solution: $\lim\limits_{x\to -1} \dfrac{x^2-1}{x+1} = \lim\limits_{x\to -1} \dfrac{(x+1)(x-1)}{(x+1)} = \lim\limits_{x\to -1}(x-1) = -1-1 = -2$

Example 2 : Compute $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$

Solution: $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{x-1} = \lim\limits_{x\to 1}(x^2+x+1)$

$$= ((1)^2+1+1) = 3$$

Example 3 : Evaluate $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-x}$

Solution: $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \dfrac{(x+2)(x-1)}{x(x-1)} = \lim_{x\to 1} \dfrac{(x+2)}{x} = \dfrac{1+2}{1} = 3$

Example 4 : Evaluate $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution : $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \dfrac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1} (\sqrt{x}+1) = \sqrt{1}+1 = 2$

3. The Conjugate Method

Example 1 : Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2}$

Solution : $\dfrac{\sqrt{x^2+100}-10}{x^2} = \dfrac{\sqrt{x^2+100}-10}{x^2} \times \dfrac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} = \frac{1}{\sqrt{x^2+100}+10}$$

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}$$

Example 2: Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

Solution : $\dfrac{\sqrt{x+2}-\sqrt{2}}{x} = \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \times \dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$= \frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \frac{1}{\sqrt{x+2}+\sqrt{2}}$$

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Example 3 : Evaluate $\lim\limits_{x\to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$

Solution : $\lim\limits_{x\to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}} = \lim\limits_{x\to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}} \times \dfrac{3+\sqrt{5+x}}{3+\sqrt{5+x}} \times \dfrac{1+\sqrt{5-x}}{1+\sqrt{5-x}}$

$$= \lim_{x\to 4} \frac{(9-5-x)\left(1+\sqrt{5-x}\right)}{(1-5+x)\left(3+\sqrt{5+x}\right)}$$

$$= \lim_{x\to 4} \frac{(4-x)\left(1+\sqrt{5-x}\right)}{(-4+x)\left(3+\sqrt{5+x}\right)}$$

$$= \lim_{x\to 4} \frac{(4-x)\left(1+\sqrt{5-x}\right)}{-(4-x)\left(3+\sqrt{5+x}\right)}$$

$$= \lim_{x\to 4} \frac{-\left(1+\sqrt{5-x}\right)}{\left(3+\sqrt{5+x}\right)} = \frac{-\left(1+\sqrt{5-4}\right)}{\left(3+\sqrt{5+4}\right)} = \frac{-2}{6} = \frac{-1}{3}$$

Evaluate 4 : $\lim\limits_{h\to 0} \dfrac{\sqrt[3]{8+h}-2}{h}$

Solution: Let $x^3 = 8+h$

When $h \to 0 \implies x^3 \to 8 \implies x \to 2$

$$\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h} = \lim_{x\to 2} \frac{\sqrt[3]{x^3}-2}{x^3-8}$$

$$= \lim_{x\to 2} \frac{x-2}{x^3-8} = \lim_{x\to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$$

$$= \lim_{x\to 2} \frac{1}{(x^2+2x+4)} = \frac{1}{((2)^2+2(2)+4)} = \frac{1}{12}$$

Evaluate 5: $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

Solution: Let $t^6 = x$

When $x \to 1 \implies t^6 \to 1 \implies t \to 1$

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{\sqrt[3]{t^6}-1}{\sqrt{t^6}-1}$$

$$= \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)(t^2+t+1)}$$

$$= \lim_{t\to 1} \frac{t+1}{t^2+t+1} = \frac{2}{3}$$

Homework.

q❶/ Evaluate the following limits

(1)	$\lim_{x\to -1}(4-2x-x^2)$	5	(2)	$\lim_{x\to 0}\dfrac{3^x-3^{-x}}{3^x+3^{-x}}$	$\dfrac{4}{5}$
(3)	$\lim_{x\to 3}\dfrac{x^2+x-1}{x^2+3x}$	$\dfrac{11}{18}$	(4)	$\lim_{x\to 0}\dfrac{4^{2t}-1}{4^t-1}$	2
(5)	$\lim_{x\to 3/2}\dfrac{4x^2-9}{2x-3}$	6	(6)	$\lim_{h\to 4}\dfrac{(h+2)^2-9h}{h-4}$	3
(7)	$\lim_{x\to 1}\sqrt[3]{\dfrac{64x^3+3x-3}{2x^{11}+3x^2+3x}}$	2	(8)	$\lim_{t\to -2}\dfrac{2t+4}{12-3t^2}$	$\dfrac{1}{6}$
(9)	$\lim_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$	-1	(10)	$\lim_{x\to 7}\dfrac{\sqrt{(t-7)^3}}{t-7}$	0
(11)	$\lim_{h\to 0}\dfrac{(3+h)^2-9}{h}$	6	(12)	$\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$	2
(13)	$\lim_{t\to 1}\dfrac{1-t^3}{1-t^2}$	$\dfrac{3}{2}$	(14)	$\lim_{x\to 4}\left(\dfrac{1}{\sqrt{x}-2}-\dfrac{4}{x-4}\right)$	$\dfrac{1}{4}$
(15)	$\lim_{x\to 0}\dfrac{1}{x}\left(1-\dfrac{1}{(x+1)^2}\right)$	2	(16)	$\lim_{t\to 2}\dfrac{2^{2t}+2^t-20}{2^t-4}$	9
(17)	$\lim_{x\to 0}\dfrac{\sqrt{7-x}-\sqrt{7}}{x}$	$\dfrac{-1}{2\sqrt{7}}$	(18)	$\lim_{h\to 0}\dfrac{\sqrt[4]{1+h}-1}{h}$	$\dfrac{1}{4}$
(19)	$\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$	$\dfrac{1}{6}$	(20)	$\lim_{h\to 0}\dfrac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$	$\dfrac{2}{3}$
(21)	$\lim_{x\to 1}\dfrac{\sqrt{4-x^2}}{2+x}$	$\dfrac{1}{\sqrt{3}}$	(22)	$\lim_{x\to 3}\dfrac{x^4-18x^2+81}{(x-3)^2}$	36
(23)	$\lim_{x\to -1}[(x+4)^3.(x+2)^{-1}]$	27	(24)	$\lim_{u\to 1}\dfrac{(3u+4)(2u-2)^2}{(u-1)^2}$	0

(25)	$\lim\limits_{s\to 1/2}\dfrac{s+4}{2s}$	$\dfrac{9}{2}$	(26)	$\lim\limits_{x\to 3}\dfrac{x^3+9x^2+20x}{x^2+x-12}$	∞
(27)	$\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$	$2x$	(28)	$\lim\limits_{a\to b}\dfrac{a^2-3ab+2b^2}{a-b}$	$-2b$
(29)	$\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$	$\dfrac{3}{2}$	(30)	$\lim\limits_{x\to 0}\left(\dfrac{1}{3x}-\dfrac{1}{x(x+3)}\right)$	$\dfrac{1}{9}$
(31)	$\lim\limits_{t\to -2}\dfrac{t^3+4t^2+4t}{(t+2)(t-3)}$	0	(32)	$\lim\limits_{x\to 1}\dfrac{x-1}{x^3-x^2+x-1}$	$\dfrac{1}{2}$
(33)	$\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{3x^2-5x-2}$	1	(34)	$\lim\limits_{x\to 7}\dfrac{x-7}{x^3-7x^2+7x-49}$	$\dfrac{1}{56}$
(35)	$\lim\limits_{x\to a}\dfrac{x^2+(1-a)x-a}{x-a}$	$a+1$	(36)	$\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-x}{x-1}$	$\dfrac{-2}{3}$
(37)	$\lim\limits_{x\to 4}\dfrac{3x^2-17x+20}{4x^2-25x+36}$	1	(38)	$\lim\limits_{t\to 2}\dfrac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2}$	$\dfrac{\sqrt{6}}{9}$
(39)	$\lim\limits_{h\to 0}\dfrac{(2+h)^4-16}{h}$	32	(40)	$\lim\limits_{x\to -\frac{3}{4}}\dfrac{16\,x^2-9}{4x+3}$	-6
(41)	$\lim\limits_{t\to 0}\dfrac{\sqrt{25+6t}-5}{t}$	$\dfrac{3}{10}$	(42)	$\lim\limits_{x\to \frac{1}{3}}\dfrac{1-9x^2}{1-3x}$	2
(43)	$\lim\limits_{h\to 0}\dfrac{\sqrt[3]{1+2x}-1}{x}$	$\dfrac{2}{3}$	(44)	$\lim\limits_{x\to -1}\dfrac{2x^2-x-3}{3x^2+8x+5}$	$-\dfrac{5}{2}$
(45)	$\lim\limits_{h\to 1}\dfrac{\sqrt{h}-1}{h-1}$	$\dfrac{1}{2}$	(46)	$\lim\limits_{x\to -1}\dfrac{2\sqrt{x}-6}{x-9}$	$\dfrac{1}{3}$
(47)	$\lim\limits_{h\to -4}\dfrac{\sqrt{2(h^2-8)}+h}{h+4}$	-1	(48)	$\lim\limits_{x\to 8}\dfrac{\sqrt{7+\sqrt[3]{x}}-3}{x-8}$	$\dfrac{1}{72}$
(49)	$\lim\limits_{h\to 0}\dfrac{(2+h)^{-3}-2^{-3}}{h}$	$\dfrac{-3}{16}$	(50)	$\lim\limits_{x\to 1}\dfrac{1}{x-1}\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$	$\dfrac{1}{32}$
(51)	$\lim\limits_{x\to a}\dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}$	$\dfrac{1}{3\sqrt[3]{a^2}}$	(52)	$\lim\limits_{x\to 3}\dfrac{1}{x-3}\left(\dfrac{1}{\sqrt{x+1}}-\dfrac{1}{2}\right)$	$\dfrac{-1}{16}$
(53)	$\lim\limits_{t\to 2}\dfrac{\sqrt{(t+4)(t-2)^4}}{(3t-6)^2}$	$\dfrac{\sqrt{6}}{9}$	(54)	$\lim\limits_{x\to 81}\dfrac{\sqrt[4]{x}-3}{x-81}$	$\dfrac{1}{108}$

(55)	$\lim\limits_{t\to 7} \dfrac{\sqrt{(t-7)^3}}{t-7}$	0	(56)	$\lim\limits_{p\to 1} \dfrac{p^5-1}{p-1}$	5
(57)	$\lim\limits_{x\to 3} \dfrac{x^4-18x^2+81}{(x-3)^2}$	36	(58)	$\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$	80
(59)	$\lim\limits_{u\to 1} \dfrac{(3u+4)(2u-2)^3}{(u-1)^2}$	-56	(60)	$\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{x-1}$	$\dfrac{1}{3}$
(61)	$\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$	$\dfrac{1}{32}$	(62)	$\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$	$\dfrac{1}{32}$
(63)	$\lim\limits_{x\to \pi} \dfrac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2}$	-1	(64)	$\lim\limits_{b\to 2} \dfrac{3b}{\sqrt{4b+1}-1}$	3
(65)	$\lim\limits_{x\to 2} \dfrac{1-2/x}{x^2-4}$	$\dfrac{1}{8}$	(66)	$\lim\limits_{x\to b} \dfrac{(x-b)^{50}-x+b}{x-b}$	-1
(67)	$\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$	4	(68)	$\lim\limits_{h\to 0} \left(\dfrac{1-h^2}{h^2}+\dfrac{6h^2-1}{h^2}\right)$	5
(69)	$\lim\limits_{x\to 1} \dfrac{\sqrt{1-x^2}}{1-x}$	∞	(70)	$\lim\limits_{x\to b} \dfrac{(2x-1)^2-9}{x+1}$	-12
(71)	$\lim\limits_{x\to 0} \dfrac{\sqrt{9+5x+4x^2}-3}{x}$	$\dfrac{5}{6}$	(72)	$\lim\limits_{w\to -k} \dfrac{w^2+6kw-4k^2}{w^2+kw}$	-3
(73)	$\lim\limits_{x\to 2} \dfrac{\sqrt[3]{10-x}-2}{x-2}$	$-\dfrac{1}{12}$	(74)	$\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$	$5a^4$
(75)	$\lim\limits_{x\to 2} \dfrac{\sqrt{4-4x+x^2}}{x-2}$	-1	(76)	$\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$	7
(77)	$\lim\limits_{x\to 0} \dfrac{t^2+3t}{(t+2)^2-(t-2)^2}$	$\dfrac{3}{8}$	(78)	$\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1}$	6
(79)	$\lim\limits_{y\to 1} \dfrac{y-4\sqrt{y}+3}{y^2-1}$	$-\dfrac{1}{2}$	(80)	$\lim\limits_{x\to 2} \dfrac{1-\dfrac{2}{x}}{x^4-4}$	$\dfrac{1}{8}$
(81)	$\lim\limits_{x\to 8} \dfrac{x^{2/3}-4}{x^{1/3}-2}$	4	(82)	$\lim\limits_{x\to 1} \dfrac{x^3-1}{\sqrt{2x+2}-2}$	6

2.2 Limits at Infinity

Infinity is a very special idea. We know we can't reach it, but we can still try to work out the value of functions that have infinity in them.

The indeterminate expressions are

$$\boxed{\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^0, \infty^0, 1^\infty}$$

Note: When the limit is fractional and x approaches from ∞ we must divided on the highest power of x in the problem.

Example 1: Compute $\lim\limits_{x \to \infty} \dfrac{x}{1 + x^2}$

Solution: $$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^2}+1} = \frac{\frac{1}{\infty}}{\frac{1}{(\infty)^2}+1} = \frac{0}{0+1} = 0$$

Example 2: Compute $\lim\limits_{x \to -\infty} \dfrac{2x^3}{1 + x^3}$

Solution: $$\lim_{x\to-\infty} \frac{2x^3}{1+x^3} = \lim_{x\to\infty} \frac{\frac{2x^3}{x^3}}{\frac{1}{x^3}+\frac{x^3}{x^3}} = \lim_{x\to\infty} \frac{2}{\frac{1}{x^3}+1} = \frac{2}{\frac{1}{(-\infty)^3}+1} = \frac{2}{0+1} = 2$$

Example 3: Compute $\lim\limits_{x \to -\infty} \dfrac{2x^3 - 3x + 5}{4x^5 - 2}$

Solution: $$\lim_{x\to-\infty} \frac{2x^3-3x+5}{4x^5-2} = \lim_{x\to\infty} \frac{\frac{2x^3}{x^5}-\frac{3x}{x^5}+\frac{5}{x^5}}{\frac{4x^5}{x^5}-\frac{2}{x^5}} = \lim_{x\to\infty} \frac{\frac{2}{x^2}-\frac{3}{x^4}+\frac{5}{x^5}}{4-\frac{2}{x^5}} = \frac{0}{4} = 0$$

Example 4: Compute $\lim\limits_{x\to\infty} \dfrac{2x+5}{\sqrt{2x^2-5}}$

Solution: $\lim\limits_{x\to\infty} \dfrac{2x+5}{\sqrt{2x^2-5}} = \lim\limits_{x\to\infty} \dfrac{\frac{2x}{x}+\frac{5}{x}}{\sqrt{\frac{2x^2}{x^2}-\frac{5}{x^2}}} = \lim\limits_{x\to\infty} \dfrac{2+\frac{5}{x}}{\sqrt{2-\frac{5}{x^2}}} = \dfrac{2+0}{\sqrt{2-0}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

Example 5: $\lim\limits_{x\to\infty} \dfrac{2x}{\sqrt{x+2}+\sqrt{x}}$

Solution: $\lim\limits_{x\to\infty} \dfrac{2x}{\sqrt{x+2}+\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{\frac{2x}{x}}{\frac{\sqrt{x+2}}{x}+\frac{\sqrt{x}}{x}} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{\frac{x+2}{x^2}}+\sqrt{\frac{x}{x^2}}}$

$$\lim_{x\to\infty} = \frac{2}{\sqrt{\frac{x}{x^2}+\frac{2}{x^2}}+\sqrt{\frac{x}{x^2}}} = \lim_{x\to\infty} \frac{2}{\sqrt{0+0}+0} = \frac{2}{0} = \infty$$

Example 6: Compute $\lim\limits_{x\to\infty} \sqrt{\dfrac{x+1}{x+2}}$

Solution: $\lim\limits_{x\to\infty} \sqrt{\dfrac{x+1}{x+2}} = \sqrt{\lim\limits_{x\to\infty} \dfrac{x+1}{x+2}} = \sqrt{\lim\limits_{x\to\infty} \dfrac{\frac{x}{x}+\frac{1}{x}}{\frac{x}{x}+\frac{2}{x}}} = \sqrt{\lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{1+\frac{2}{x}}}$

$$= \sqrt{\frac{1+\frac{1}{\infty}}{1+\frac{2}{\infty}}} = \sqrt{\frac{1+0}{1+0}} = \sqrt{1} = 1$$

Example 7: Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2+1}{x^{3/2}}$

Solution: $\lim\limits_{x\to\infty} \dfrac{x^2+1}{x^{3/2}} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^{3/2}}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x^2}}{\frac{1}{x^{1/2}}} = \dfrac{1}{0} = \infty$

Example 8: Evaluate $\lim\limits_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1}$

Solution: $\lim\limits_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1} = \lim\limits_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1} \times \dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$

$$\lim_{x\to\infty} \frac{(x^2+1)-(x^2-1)}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}} = 0$$

Example 9: Evaluate $\lim\limits_{x\to\infty} \left(\sqrt{3x^2+2x+1} - \sqrt{2}\,x\right)$

Solution: $\lim\limits_{x\to\infty} \sqrt{3x^2+2x+1} - \sqrt{2}\,x \times \dfrac{\sqrt{3x^2+2x+1}+\sqrt{2}\,x}{\sqrt{3x^2+2x+1}+\sqrt{2}\,x}$

$$\lim_{x\to\infty} \frac{3x^2+2x+1-2x^2}{\sqrt{3x^2+2x+1}+\sqrt{2}\,x} = \lim_{x\to\infty} \frac{x^2+2x+1}{\sqrt{3x^2+2x+1}+\sqrt{2}\,x}$$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{\sqrt{\frac{3x^2}{x^4}+\frac{2x}{x^4}+\frac{1}{x^4}}+\frac{\sqrt{2}\,x}{x^2}} = \frac{1+0+0}{\sqrt{0+0+0}+0} = \frac{1}{0} = \infty$$

Note: We can use the following rules when the limit is rational and approaches from infinity

❶ When the degree of the numerator is less than the denominator.

The product is zero

$$\lim_{x\to\infty} \frac{x^2-3x}{4x^3-2} = 0$$

❷ When the degree of the numerator is the same as the denominator

The product is coefficient of higher x in numerator to coefficient of higher x in denominator.

$$\lim_{x\to\infty} \frac{2x^3}{3x^3+1} = \frac{2}{3}$$

❸ When the degree of the numerator is greater than the denominator

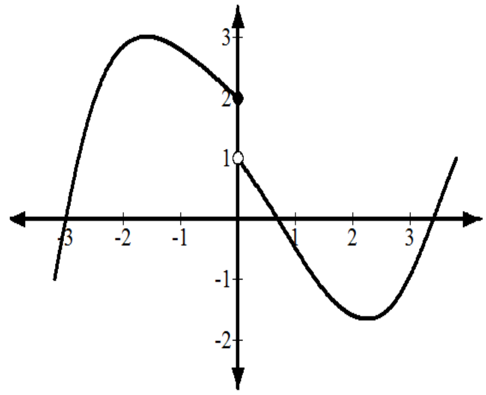
The product is infinity

$$\lim_{x\to\infty} \frac{x^5+1}{x^3} = \infty$$

2.3 One – Sided Limits

Suppose that a function f has the graph shown on the right. Notice that $f(0) = 2$. (Function value at 0) If $x \approx 0$ *and* $x < 0$, *then* $f(x) \approx 2$.

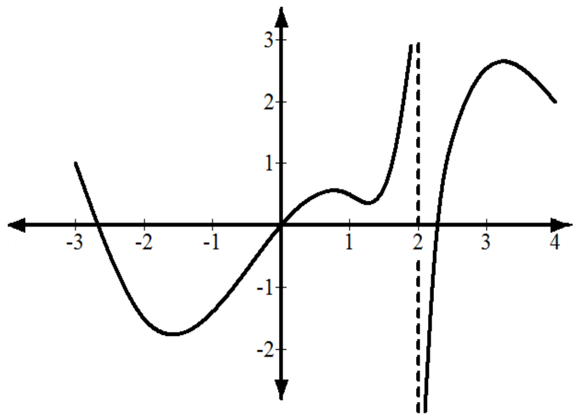
But if $x \approx 0$ and $x > 0$, then $f(x) \approx 1$Therefore, the limit of f(x) as x approaches 0 does not exist.



$\lim\limits_{x \to a} f(x)$ exists if and only if

$\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist and are equal.

Suppose that a function f has the graph shown on the right. Here the interesting behavior of f is in vicinity of x=2.



Notes that $f(2)$ is undefined an line $x = 2$ is a vertical asymptote.

If $x \approx 2$ and $x < 2$ then $f(x)$ is large and positive.

But if $x \approx 2$ and $x > 2$ then $f(x)$ is large and negative.

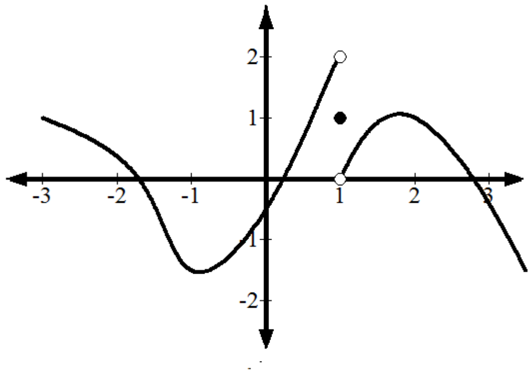
Therefore, the limit of $f(x)$ as x approaches 2 does not exist. In fact, neither of the one-sided limits exists. However, we can describe the nature of the vertical asymptote by writing

$$\lim_{x \to 2^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = -\infty$$

Example1: Given the function f whose graph is below, determine the following:

[a] $f(1)$ [b] $\lim\limits_{x\to1^-} f(x)$ [c] $\lim\limits_{x\to1^+} f(x)$ [d] $\lim\limits_{x\to1} f(x)$

Solution:



[a] $f(1) = 1$

[b] $\lim\limits_{x\to1^-} f(x) = 2$

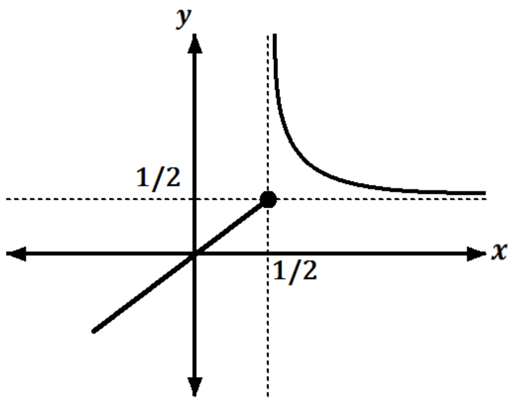
[c] $\lim\limits_{x\to1^+} f(x) = 0$

[d] $\lim\limits_{x\to1} f(x)$ does not exist

Example2: Given the function f whose graph is below, determine the following:

[a] $\lim\limits_{x\to\frac{1}{2}^+} f(x)$ [b] $\lim\limits_{x\to\frac{1}{2}^-} f(x)$ [c] $\lim\limits_{x\to\frac{1}{2}} f(x)$ [d] $\lim\limits_{x\to+\infty} f(x)$ [e] $\lim\limits_{x\to-\infty} f(x)$

Solution



[a] $\lim\limits_{x\to1/2^+} f(x) = +\infty$

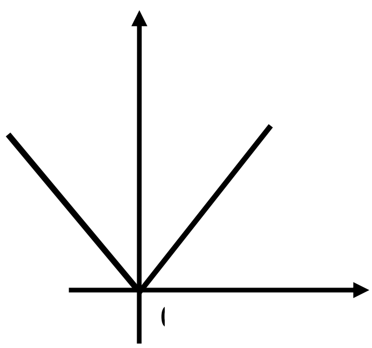
[b] $\lim\limits_{x\to1/2^-} f(x) = \dfrac{1}{2}$

[c] $\lim\limits_{x\to1/2} f(x) =$ does not exist

[d] $\lim\limits_{x\to+\infty} f(x) = \dfrac{1}{2}$

[e] $\lim\limits_{x\to-\infty} f(x) = -\infty$

Example 3: Show that $\lim\limits_{x\to0} |x| = 0$ exist



Solution/ $\lim\limits_{x\to0^+} |x| = \lim\limits_{x\to0^+} x = 0 \quad$ for $x > 0$

$\lim\limits_{x\to0^-} |x| = \lim\limits_{x\to0^-} (-x) = 0 \quad$ for $x < 0$

$\therefore \lim\limits_{x\to0} |x| = 0$

This function is exist

Example 4: f(x)= $\begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8-2x & \text{if} \quad x < 4 \end{cases}$ Determine where there $\operatorname{Lim}_{x\to 4}$ exist

Solution : $f(x) = \sqrt{x-4}$ for x > 4 right side

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = \mathbf{0}$$

$$f(x) = 8-2x \qquad \text{for } x < 4 \text{ left side}$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 8-2x = \sqrt{4-4} = \mathbf{0}$$

Example 5: if $f(x) = \dfrac{|x-2|}{x^2+x-6}$, determine the following: –

$\boxed{a}$ $\lim_{x\to 2^+} f(x)$ $\boxed{b}$ $\lim_{x\to 2^-} f(x)$ $\boxed{c}$ $\lim_{x\to 2} f(x)$

$\boxed{a}$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x-2}{x^2+x-6} = \lim_{x\to 2^+} \dfrac{x-2}{(x+3)(x-2)} = \lim_{x\to 2^+} \dfrac{1}{x+3} = \dfrac{1}{5}$

$\boxed{b}$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{-(x-2)}{x^2+x-6} = \lim_{x\to 2^-} \dfrac{-(x-2)}{(x+3)(x-2)} = \lim_{x\to 2^-} \dfrac{-1}{x+3} = -\dfrac{1}{5}$

$\boxed{c}$ $\lim_{x\to 2} f(x) =$ does not exist

2.4 Continuity

Continuity of a graph is loosely defined as the ability to draw a graph without having to lift your pencil.

If $f(x)$ is defined on an open interval containing c, then $f(x)$ is said to be continuous at c if and only if the limit as x approaches c equals $f(x)$

$$\lim_{x\to c} f(x) = f(c)$$

✧ Can we say the function f is continuous at a number c if:-

1. $f(c)$ is defined (c is in the domain of f)
2. $\lim_{x\to c} f(x)$ exist
3. $\lim_{x\to c} f(x) = f(c)$

Example1: given $f(x) = \begin{cases} x^2+3 & x \le 2 \\ 3x+2 & x > 2 \end{cases}$ is the function continuous at $x = 2$?

Solution:

① At $x = 2$ $f(x) = x^2 + 3$ $f(2) = (2)^2 + 3 = \boxed{7}$

② $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 + 3) = \boxed{7}$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3x + 2) = \boxed{6}$

Therefore the function is not continuous at $x = 2$

Example2: $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 0 < x \le 2 \end{cases}$ is the function continuous at $x = 1$?

Solution:

① $At\ x = 1\ f(x) = x$ $f(1) = 1$

② $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$ $\lim\limits_{x \to 1} f(x) = 1$

③ $f(1) = \lim\limits_{x \to 1} f(x)$

The function is continuous

Example3: What value should be assigned to (a) to make the function continuous at $x = 3$?

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \ge 3 \end{cases}$$

Solution:

Because the function is continuous at $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} (x^2 - 1) = \lim_{x \to 3^+} (2ax)$$

$$9 - 1 = 6a \Longrightarrow a = \frac{8}{6} = \boxed{\frac{4}{3}}$$

Example 4: What value should be assigned to (a) to make the function continuous at $x = -1$?

$$f(x) = \begin{cases} x + 2p & x \leq -1 \\ p^2 & x > -1 \end{cases}$$

Solution:

Because the function is continuous at $x = -1$

$$\lim_{x\to-1^-} (x + 2p) = \lim_{x\to-1^+} p^2$$

$$-1 + 2p = p^2$$

$$p^2 - 2p + 1 = 0$$

$$((p-1)(p-1) = 0 \Longrightarrow p = \boxed{1}$$

Example 5: Determine the values of A and B if the function is continuous at $x = 2 \; and \; x = 5$?

$$f(x) = \begin{cases} 6x & x \leq 2 \\ Ax + B & 2 < x < 5 \\ -3x & x \geq 5 \end{cases}$$

Solution:

Because the function is continuous at $x = 2$

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$$

$$\lim_{x\to2^-} (6x) = \lim_{x\to2^+} (Ax + B)$$

$$12 = 2A + B \Longrightarrow 2A + B = 12 \quad \ldots\ldots\ldots..(1)$$

Because the function is continuous at $x = 5$

$$\lim_{x\to5^-} f(x) = \lim_{x\to5^+} f(x)$$

$$\lim_{x\to5^-} (Ax + B) = \lim_{x\to5^+} (-3x)$$

$$5A + B = -15 \quad \ldots\ldots\ldots..(2)$$

By solving the equations (1) and (2) numerically ,we obtain

$$A = \boxed{-9} \text{ and } B = \boxed{30}$$

Homework.

q1/ Evaluate the following limits

(1)	$\lim\limits_{x\to\infty}\dfrac{t^2}{7-t^2}$	-1	(2)	$\lim\limits_{x\to\infty}\dfrac{2x+1}{\sqrt{x^2+3}}$	2
(3)	$\lim\limits_{x\to\infty}\dfrac{x^2}{(x-5)(3-x)}$	-1	(4)	$\lim\limits_{t\to\infty}\left(\dfrac{1}{3t^2}-\dfrac{5t}{t+2}\right)$	-5
(5)	$\lim\limits_{x\to\infty}\dfrac{x^3}{2x^3-100x^2}$	$\dfrac{1}{2}$	(6)	$\lim\limits_{x\to\infty}\dfrac{n^2}{\sqrt{n^3+2n}}$	∞
(7)	$\lim\limits_{x\to\infty}\dfrac{3\sqrt{x^3}+3x}{\sqrt{2x^3}}$	$\dfrac{3}{\sqrt{2}}$	(8)	$\lim\limits_{x\to\infty}x\left(\sqrt{x^2-1}-x\right)$	$-\dfrac{1}{2}$
(9)	$\lim\limits_{s\to\infty}\dfrac{8-s}{\sqrt{s^2+5}}$	-1	(10)	$\lim\limits_{x\to\infty}\left(\dfrac{x-4(x+1)^2}{(x+1)^2}\right)$	-4
(11)	$\lim\limits_{x\to\infty}\dfrac{\sqrt{2x^2-3}}{x+3}$	$\sqrt{2}$	(12)	$\lim\limits_{x\to\infty}\left(2-\dfrac{2x^2}{(x+1)^2}\right)$	0
(13)	$\lim\limits_{x\to\infty}\dfrac{\sqrt{6x^6-x}}{x^3+1}$	$\sqrt{6}$	(14)	$\lim\limits_{t\to 2}\dfrac{t+2}{(t-2)^2}$	∞
(15)	$\lim\limits_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$	$\dfrac{a-b}{2}$	(16)	$\lim\limits_{x\to\infty}\sqrt[3]{\dfrac{\pi x^3+3x}{\sqrt{2}\,x^3+7x}}$	$\sqrt[3]{\dfrac{\pi}{\sqrt{2}}}$
(17)	$\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x}$	2	(18)	$\lim\limits_{x\to\infty}\dfrac{3x^3-x^2}{\pi x^3-5x^2}$	$\dfrac{3}{\pi}$
(19)	$\lim\limits_{x\to\infty}\left(\sqrt{2x^2+3}-\sqrt{2x^2-5}\right)$	0	(20)	$\lim\limits_{x\to\infty}\dfrac{3\sqrt{x^3}+3x}{\sqrt{2x^3}}$	$\dfrac{3}{\sqrt{2}}$
(21)	$\lim\limits_{x\to\infty}\sqrt[3]{\dfrac{1+8x^2}{x^2+4}}$	2	(22)	$\lim\limits_{x\to\infty}\left(\dfrac{20x^2-3x}{3x^5-4x^2+5}\right)$	0
(23)	$\lim\limits_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2+3x}\right)$		(24)	$\lim\limits_{x\to\infty}\dfrac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}}$	∞

q❷/ if $f(x) = 2 + |5x - 1|$, determine the following:-

$\boxed{a}$ $\lim\limits_{x \to \frac{1}{5}^{+}} f(x)$ Sol. : 2

$\boxed{b}$ $\lim\limits_{x \to \frac{1}{5}^{-}} f(x)$ Sol. : 2

$\boxed{c}$ $\lim\limits_{x \to \frac{1}{5}} f(x)$ Sol. : 2

q❸/ if $f(x) = \begin{cases} \dfrac{1}{x} & x < 0 \\ x^2 & 0 \le x < 1 \\ 2 & x = 1 \\ 2 - x & x > 1 \end{cases}$

Determine the following:-

$\boxed{a}$ $\lim\limits_{x \to -1} f(x)$ Sol. -1

$\boxed{b}$ $\lim\limits_{x \to 1} f(x)$ Sol. 1

$\boxed{c}$ $\lim\limits_{x \to 0^{+}} f(x)$ Sol. 0

$\boxed{d}$ $\lim\limits_{x \to 0^{-}} f(x)$ Sol. $-\infty$

$\boxed{e}$ $\lim\limits_{x \to 0} f(x)$ Does not exist

$\boxed{f}$ $\lim\limits_{x \to 2^{+}} f(x)$ Sol.0

$\boxed{g}$ $\lim\limits_{x \to 2^{-}} f(x)$ Sol.0

$\boxed{h}$ $\lim\limits_{x \to 2} f(x)$ Sol.0

q❹: - Given the function f whose graph is below, determine the following:-

$\lim\limits_{x \to -3} f(x)$ Sol. 2

$f(-3)$ Sol.

$f(-1)$ Sol.

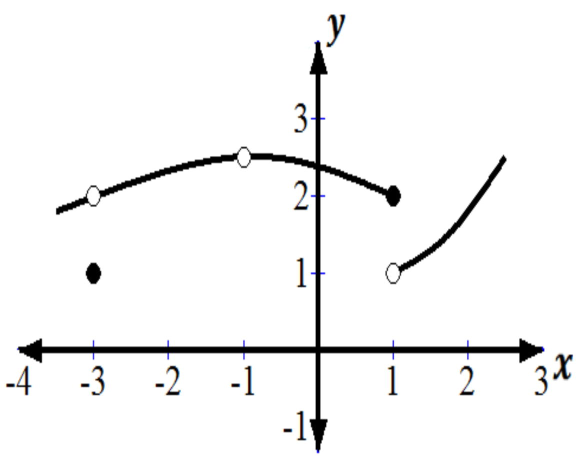
$\lim\limits_{x \to -1} f(x)$ Sol.

$f(1)$ Sol.

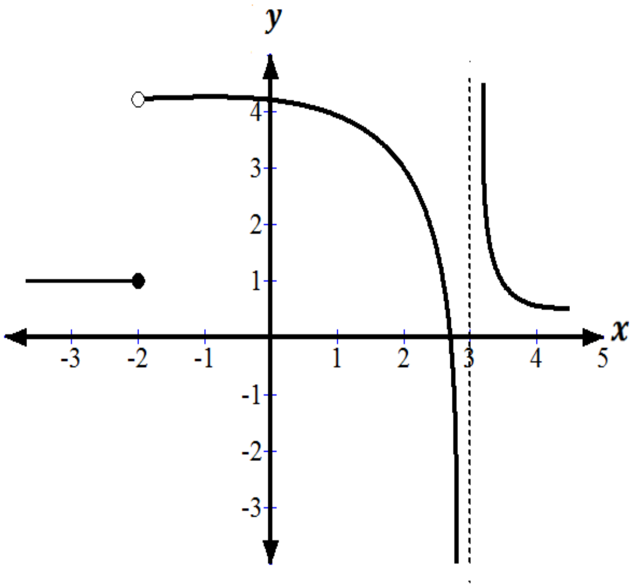
$\lim\limits_{x \to 1} f(x)$ Sol.

$\lim\limits_{x \to 1^{+}} f(x)$ Sol.

$\lim\limits_{x \to 1^{+}} f(x)$ Sol.



q❺: - Given the function f whose graph is below, determine the following:-



a	$f(-2)$	*Sol.*	1
b	$f(3)$	*Sol.*	un defined
c	$\lim\limits_{x\to -2^-}$	*Sol.*	1
d	$\lim\limits_{x\to -2^+}$	*Sol.*	4
e	$\lim\limits_{x\to -2}$	*Sol.*	Not exist
f	$\lim\limits_{x\to 3^+}$	*Sol.*	$+\infty$
f	$\lim\limits_{x\to 3^-}$	*Sol.*	$-\infty$

q❻ Show that the limit $\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$ do not exist.

q❼ If $f(x)=\begin{cases} x^2+3 & \text{If} \quad x\le 1 \\ x+1 & \text{If} \quad x>1 \end{cases}$

Find (a) $\lim\limits_{x\to 1^+} f(x)$ (b) $\lim\limits_{x\to 1^-} f(x)$

Answer: (a) 2 (b) 4

q❽ If $f(x)=\begin{cases} x^2+1 & \text{If} \quad x<-1 \\ \sqrt{x+1} & \text{If} \quad x\ge -1 \end{cases}$

Find (a) $\lim\limits_{x\to -1^+} f(x)$ (b) $\lim\limits_{x\to -1^-} f(x)$ (c) $\lim\limits_{x\to -1} f(x)$

Answer: (a) 0 (b)2 (c) DNE

q❾ If $f(x)=\begin{cases} 3 & \text{If} \quad x\le -2 \\ -1/2\,x^2 & \text{If} \quad -2<x<2 \\ 3 & \text{If} \quad x\ge 2 \end{cases}$

Find (a) $\lim\limits_{x\to \pm 2^+} f(x)$ (b) $\lim\limits_{x\to \pm 2^-} f(x)$

Answer: (a) -2 ,3 (b) 3, -2

q❿ If $f(x)=\begin{cases} x^2-4x+5 & \text{If} \quad x<2 \\ 4-x & \text{If} \quad x\ge 2 \end{cases}$

Find (a) $\lim\limits_{x\to 2^+} f(x)$ (b) $\lim\limits_{x\to 2^-} f(x)$ (c) $\lim\limits_{x\to -2} f(x)$ (d) $f(2)$

Answer: (a) 2 (b)1 (c) DNE (d)2

q11/ If $f(x) = \begin{cases} x+2 & \text{If} \quad x \le -1 \\ ax^2 & \text{If} \quad x > -1 \end{cases}$

Find a so that $\lim\limits_{x\to -1} f(x)$ exists   Answer: $a = 1$

q12/ If $f(x) = \begin{cases} 2x+3 & \text{If} \quad x < 1 \\ 2 & \text{If} \quad x = 1 \\ 7-2x & \text{If} \quad 1 < x \end{cases}$

Find (a) $\lim\limits_{x\to 1^+} f(x)$ (b) $\lim\limits_{x\to 1^-} f(x)$ (c) $\lim\limits_{x\to 1} f(x)$

Answer: (a) 5 (b) 5 (c) 5

q13/ If $f(x) = \begin{cases} \sqrt{x^2-9} & \text{If} \quad x \le -3 \\ \sqrt{9-x^2} & \text{If} \quad -3 < x < 3 \\ \sqrt{x^2-9} & \text{If} \quad 3 \le x \end{cases}$

Find (a) $\lim\limits_{x\to -3^-} f(x)$ (b) $\lim\limits_{x\to -3^+} f(x)$ (c) $\lim\limits_{x\to -3} f(x)$

(d) $\lim\limits_{x\to 3^-} f(x)$ (e) $\lim\limits_{x\to 3^+} f(x)$ (f) $\lim\limits_{x\to 3} f(x)$

Answer: (a) 0 (b) 0 (c) 0 (d) 0 (e) 0 (f) 0

q14/ If $f(x) = \begin{cases} 3x+2 & \text{If} \quad x < 4 \\ 5x+k & \text{If} \quad x \ge 4 \end{cases}$

Find k so that $\lim\limits_{x\to 4} f(x)$ exists   Answer: $k = -6$

q15/ If $f(x) = \begin{cases} kx-3 & \text{If} \quad x \le -1 \\ x^2+k & \text{If} \quad x > -1 \end{cases}$

Find k so that $\lim\limits_{x\to -1} f(x)$ exists   Answer: $a = -2$

q16/ If $f(x) = \begin{cases} x^2 & \text{If} \quad x \le -2 \\ ax+b & \text{If} \quad -2 < x < 2 \\ 2x-6 & \text{If} \quad 2 \le x \end{cases}$

Find a *and* b so that $\lim\limits_{x\to -2} f(x)$ and $\lim\limits_{x\to 2} f(x)$ are exist

Answer: $a = -\frac{3}{2}, b = 1$

q17 $f(x)=\begin{cases} x-4 & \text{If} \quad -1<x\leq 2 \\ x^2-6 & \text{If} \quad 2<x<5 \end{cases}$ is the function continuous at x=2 ?

Answer: The function is continuous at x=2

q18 / $f(x)=\begin{cases} \dfrac{x^3-27}{x^2-9} & \text{If} \quad x\neq 3 \\ 6 & \text{If} \quad x=3 \end{cases}$ is the function continuous at $x=3$?

Answer: The function is discontinuous at x=3

q19 / $f(x)=\begin{cases} \dfrac{x^2}{\mathrm{a}}-\mathrm{a} & \text{if} \quad 0<\mathrm{x}<\mathrm{a} \\ 0 & \text{if} \quad \mathrm{x}=\mathrm{a} \\ \mathrm{a}-\dfrac{\mathrm{a}^2}{x} & \text{if} \quad x>a \end{cases}$ is the function continuous at $x=\mathrm{a}$?

Answer: The function is continuous at x=a

q20 / Find c such that the function

$$f(x)=\begin{cases} \dfrac{1-\sqrt{x}}{x-1} & \text{if} \quad 0\leq \mathrm{x}<1 \\ \mathrm{c} & \text{if} \quad x=1 \end{cases}$$

is continuous for all $x\in[0,1]$? $Answer: c=-\dfrac{1}{2}$

q21 / Find a and b such that the function

$$f(x)=\begin{cases} 2x+1 & \text{If} \quad x\leq 3 \\ ax+b & \text{If} \quad 3<x<5 \\ x^2+2 & \text{If} \quad 5\leq x \end{cases}$$

is continuous every where ?

$Answer: a=10 \ , b=-23$

q22 /Find a and b such that the function

$$f(x)=\begin{cases} x+1 & \text{If} \quad x<1 \\ ax+b & \text{If} \quad 1\leq x\leq 2 \\ 3x & \text{If} \quad x>2 \end{cases}$$

is continuous every where ?

$Answer: a=4 \ , b=-2$

q 23 find the discontinuity of the given functions:-

a $f(x) = \begin{cases} x+4 & \text{If} \quad -6 \le x < -2 \\ x & \text{If} \quad -2 \le x < 2 \\ x-4 & \text{If} \quad 2 \le x < 6 \end{cases}$

Answer: The function is discontinuous $at\ x = -2\ ,2$

b $f(x) = \begin{cases} x^3 & \text{If} \quad x < 1 \\ -4 - x^2 & \text{If} \quad 1 \le x \le 10 \\ 6x^2 + 46 & \text{If} \quad x > 10 \end{cases}$

Answer: The function is discontinuous $at\ x = 10$

c $f(x) = \begin{cases} x+2 & \text{If} \quad 0 \le x < 1 \\ x & \text{If} \quad 1 \le x < 2 \\ x+5 & \text{If} \quad 2 \le x < 3 \end{cases}$

Answer: The function is discontinuous $at\ x = 2$

q 24 / Find the values of c and d such that the function f is continuous on $[-3,3]$

3 $f(x) = \begin{cases} c & \text{If} \quad x = -3 \\ \dfrac{9 - x^2}{4 - \sqrt{x^2 + 7}} & \text{If} \quad |x| < 3 \\ d & \text{If} \quad x = 3 \end{cases}$

Answer: $c = 8\ , d = 8$

Chapter Three
Derivatives

3.1 Introducion

The derivative of a function f is that function, denoted by f', whose function (functional) value at any limit point x of the domain of the function f which is in the domain of the function f, denoted by $f'(x)$, is given by the limit of the incremental ratio as the increment in the independent variable tends to zero. That is in notation,

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

3.2 Rules of derivatives

① Constant Rule

The derivative of a constant is zero; that is, for a constant c:

$$\boxed{\frac{d}{dx}c=0}$$

Examples:

1: $y=7\Rightarrow y'=0$, 2: $y=a\Rightarrow y'=0$, 3: $y=5+\sqrt{3}\Rightarrow y'=0$

② The Power Rule

If $f(x)=x^n$, then

$$\boxed{f'(x)=\frac{d}{dx}x^n=nx^{n-1}}$$

Examples:

1: $y=x^7\ y'=7x^6$, 2: $y=x^{13}\ y'=13x^{12}$, 3: $y=x^{-5}\ y'=-5x^{-6}$

4: $y=\sqrt{x}\ y=x^{\frac{1}{2}}\ y'=\frac{1}{2}x^{-\frac{1}{2}}$ 5: $y=x^{\frac{3}{2}}\ y'=\frac{3}{2}x^{\frac{1}{2}}$

6: $y=\frac{1}{\sqrt[3]{x^5}}\Rightarrow y=\frac{1}{x^{\frac{5}{3}}}\Rightarrow y=x^{-\frac{5}{3}}\Rightarrow y'=-\frac{5}{3}x^{-\frac{8}{3}}$ 7: $y=x\Rightarrow y'=1$

③ The Constant Multiple Rule

The derivative of a constant multiplied by a function is the constant multiplied by the derivative of the original function:

$$\boxed{\frac{d}{dx}c\ f(x)=c\ f'(x)}$$

Examples:

1: $f(x)=3x^2 \Rightarrow f'(x)=3.(2x)=6x$

2: $f(x)=\frac{2}{3}x^7 \Rightarrow f'(x)=\frac{2}{3}(7x^6)=\frac{14}{3}x^6$

3: $y=-2x^4 \Rightarrow y'=-8x^3$

④ The Sum Rule

If $f(x)$, $g(x)$ and $h(x)$ are differentiable functions, then

$$\boxed{\frac{d}{dx}[\,f(x)+g(x)+h(x)]=f'(x)+g'(x)+h'(x)}$$

Examples:

1: $f(x)=5x^5-3x^2+6x+1 \Rightarrow f'(x)=25x^4-6x+6$

2: $f(x)=3x^{100}-24x^3+7x^2-x-2 \Rightarrow f'(x)=300x^{99}-72x^2+14x-11$

3: $y=\sqrt{2}x^5-\frac{x^4}{4}+5x^3+x+\pi^2 \Rightarrow y'=5\sqrt{2}x^4-\frac{1}{4}(4x^3)+15x^2+1$

$$\Rightarrow y'=5\sqrt{2}x^4-\frac{1}{4}(4x^3)+15x^2+1$$

4: $y=\frac{x^3-2x^2+x-4}{2\sqrt{x}} \Rightarrow y=\frac{x^3-2x^2+x-4}{2x^{\frac{1}{2}}}$

$$\Rightarrow y=\frac{1}{2}x^{\frac{5}{2}}-x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}-2x^{-\frac{1}{2}}$$

$$\Rightarrow y'=\frac{1}{2}\left(\frac{5}{2}x^{\frac{3}{2}}\right)-\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-2\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$\Rightarrow y=\frac{5}{4}x^{\frac{3}{2}}-\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{4}x^{-\frac{1}{2}}-x^{-\frac{3}{2}}$$

⑤ The Product Rule

If f and g are differentiable functions, then

$$\boxed{\frac{d}{dx}[\,f(x)g(x)]=f(x)g'(x)+g(x)f'(x)}$$

Examples:

1: $f(x)=(3x^2+1)(7x^3+x)$

$$f'(x)=(3x^2+1)(21x^2+1)+(7x^3+x)(6x)$$

2: $y=3x(8x^3-2)$

$$f'(x)=3x(24x^2)+\left(8x^3-2\right)(3)=72x^3+24x^3-6=96x^3-6$$

If f,g and h are differentiable functions, then

$$\boxed{\frac{d}{dx}[\,f(x)g(x)h(x)]=f(x)g(x)h'(x)+f(x)g'(x)h(x)+f'(x)g(x)h(x)}$$

Examples:

1: $f(g)=x\,y\,z \quad \Rightarrow\Rightarrow \quad f'(g)=x\,y\,z'+xy'z+x'yz$

2: $y=(2x-5)(x+2)(x^2-1)$

$$y'=(2x-5)(x+2)(2x)+(2x-5)(1)(x^2-1)+(2)(x+2)(x^2-1)$$

⑥ The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions and $g(x)\neq 0$, then

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}}$$

Example:

1: $y=\dfrac{x^2}{x^3+7} \quad \Rightarrow \quad y'=\dfrac{(x^3+7)(2x)-(x^2)(3x^2)}{(x^3+7)^2}=\dfrac{14x-x^4}{(x^3+7)^2}$

⑦ The root Rule

The derivative of a root is equal to the derivative of the radicand divided by the index times the root, with the radicand raised to the index minus one.

If f(x) and g(x) are differentiable functions and g(x)≠0, then

$$\boxed{\frac{d}{d(x)}\sqrt[k]{u}=\frac{u'}{k\sqrt[k]{u^{k-1}}}}$$

Example1:

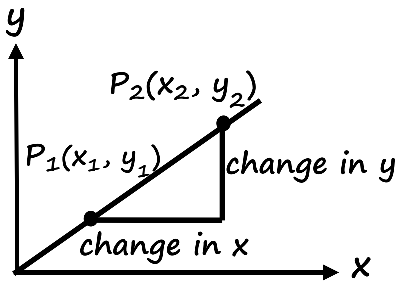
1: $y=\sqrt{5x+2} \quad \Rightarrow \quad y'=\frac{5}{2\sqrt{5x+2}}$

2: $y=\sqrt[4]{2x-4} \quad \Rightarrow \quad y'=\frac{2}{4\sqrt{(2x-4)^3}}$

2: $y=\sqrt[3]{6x^2+7x+2} \quad \Rightarrow \quad y'=\frac{12x+7}{3\sqrt{(6x^2+7x+2)^2}}$

3-3 Slopes

To calculate the slope of a line you need only two points from that line, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.



Therefore

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

While the slope for curve can be found by derivative the curve to obtain the line or the first derivative which represented the slope where this slope will cross the curve at one point only.

$$\boxed{\text{The slope}=m=f'(x)=\frac{dy}{dx}}$$

Therefore to find the slope we need the equation of curve and one point.

Example 1 : Find the slope of the line tangent of the curve $y=x^3-2x+1$ at $x=1$

Sol. $m=y'=3x^2-2$

at $x=1 \Rightarrow m=3(1)^2-2=1$

Example 2 :Find the slope of the curve at point (1,1): $y=\frac{x^3}{4}-2x+1$

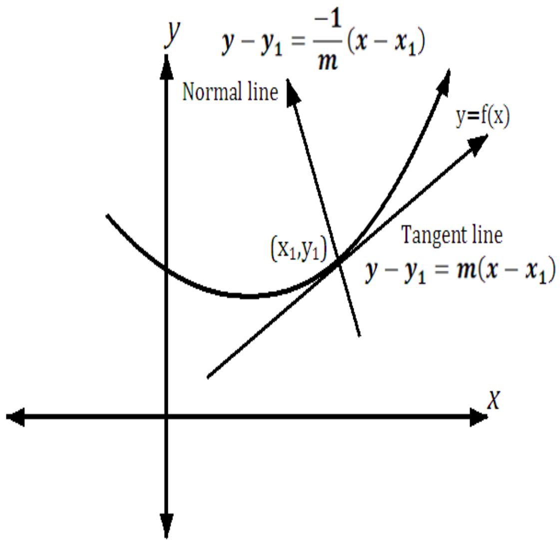
Sol. $m=y'=\frac{1}{4}3x^2-2 \Rightarrow m=\frac{3}{4}x^2-2$

at point (1,1) $\Rightarrow m=\frac{3}{4}(1)^2-2=\frac{3}{4}-2=\frac{-5}{4}$

3-4 Tangent lines

The tangent line for any curve can be found if we have slope m and point using the equation

$$\boxed{y-y_1=m(x-x_1)}$$



Notes:

1: If the line horizontal the slope is zero

2: If two lines orthogonal

(Perpendicular one to another)

$$m=\frac{1}{m_1}$$

3: If two lines parallel

$$m=m_1$$

Therefore the normal line for any curve can be found by equation

$$\boxed{y-y_1=\frac{-1}{m}(x-x_1)}$$

Example 1: Find the equation of tangent line to the curve $f(x)=4-x^2$ at the point (1,3).

Solution

$m=f'(x)=-2x$

at point (1 ,3) $\quad m=-2$

$$y-y_1=m(x-x_1)\Rightarrow y-3=-2(x-1)\Rightarrow y=-2x+5$$

Example 2: Find the equation of tangent line and normal to the curve $y=\frac{1}{x}$ at the point (1/2,2).

Solution

$m=y'=\frac{-1}{x^2}$

at point $\left(\frac{1}{2},2\right)$ $\quad m=\frac{-1}{\left(\frac{1}{2}\right)^2}=-4$

The equation of tangent line with point (1/2 ,2) and m=-4

$$y-2=-4(x-1/2)\Rightarrow y=-4x+4$$

The equation of normal line with point (1/2 ,2) and m=1/4

$$y-2=\frac{1}{4}\left(x-\frac{1}{2}\right)\Rightarrow y=\frac{x}{4}+\frac{15}{8}$$

Example 3: If $f(x)=2x^2-x$ determine the points on the curve at which the slope is parallel to $3x-y-4=0$,and then find the equation of tangent line at this point.

Solution:

The slope of line $3-y'=0 \Rightarrow y'=3 \Rightarrow m_1=3$

The slope of curve $m=m_1=3$ (parallel)

$m=f'(x)=4x-1 \Rightarrow 4x-1=3 \Rightarrow x=1 \Rightarrow y=1$

The point is (1,1)

The equation of tangent line with m=3 and point (1,1)

$$y-1=3(x-1) \Rightarrow y=3x-2$$

3-5 Implicit differentiation

Sometimes functions are given not in the form $y=f(x)$ but in a more complicated form in which it is difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions. In this unit we explain how these can be differentiated using implicit differentiation.

Example 1 : Find y' for the equation $x^2+y^2=1$

Sol. $x^2+y^2=1 \Rightarrow 2x+2yy'=0 \Rightarrow 2yy'=-2x \Rightarrow y'=\dfrac{-x}{y}$

Example 2: Find the slopes of the tangent lines to the curve $x^2+4y^2=4$ at the point $(\sqrt{2}, -1/\sqrt{2})$.

Solution

$$2x+8y\frac{dy}{dx}=0 \Rightarrow 8y\frac{dy}{dx}=-2x \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{x}{4y}=m$$

At point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$ $\qquad m=-\frac{x}{4y}=-\frac{\sqrt{2}}{4\left(-\frac{1}{\sqrt{2}}\right)}=\frac{1}{2}$

✍Example 3: Find the slopes of the tangent lines to the curve $y^2-x+1=0$ at the points (2, −1) and (2, 1).

Solution

$$2y\frac{dy}{dx}-1=0 \quad \Rightarrow \frac{dy}{dx}=\frac{1}{2y}=m$$

At point (2,−1) $\qquad m_1=\frac{1}{2(-1)}=-\frac{1}{2}$

At point (2, 1) $\qquad m_2=\frac{1}{2(1)}=\frac{1}{2}$

✍Example4: Find the tangent line to the curve $x^2(x^2+y^2)=y^2$ at the point $(\sqrt{2}/2 \ ,\sqrt{2}/2)$.

Solution

$$x^4+x^2y^2-y^2=0$$

$$4x^3+\left(x^2.2y\frac{dy}{dx}+y^2.2x\right)-2y\frac{dy}{dx}=0$$

$$2y(x^2-1)\frac{dy}{dx}=2x(2x^2+y^2)$$

$$\frac{dy}{dx}=\frac{x(2x^2+y^2)}{y(x^2-1)}$$

At point $\left(\frac{\sqrt{2}}{2} \ ,\frac{\sqrt{2}}{2}\right)$ the slope is

$$\frac{dy}{dx}=\frac{\sqrt{2}/2\left(2(\sqrt{2}/2)^2+(\sqrt{2}/2)^2\right)}{\sqrt{2}/2\left((\sqrt{2}/2)^2-1\right)}=\frac{3/2}{1/2}=3$$

The equation of the tangent line at this point is

$$y-y_1=m(x-x_1)$$

$$y-\frac{\sqrt{2}}{2}=3\left(x-\frac{\sqrt{2}}{2}\right)$$

$$y=3x-\sqrt{2}$$

✍Example 5: If the equation of the curve given by $x^3+y^3=3xy$:-

(a) Find the slope of the curve

(b) Find an equation for the tangent line to at the point $\left(\frac{3}{2},\frac{3}{2}\right)$

(c) At what point(s) in the first quadrant is the tangent line horizontal?

Solution

(a) $$3x^2+3y^2\frac{dy}{dx}=3x.\frac{dy}{dx}+3y$$

$$(y^2-x)\frac{dy}{dx}=y-x^2 \qquad \Rightarrow \frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$

(b) At the point (3/2 , 3/2) the slope m of the tangent line at this point is

$$m=\frac{dy}{dx}=\frac{(3/2)-(3/2)^2}{(3/2)^2-(3/2)}=-1$$

The equation of the tangent line at the point (3/2, 3/2) and m=-1

$$y-\frac{3}{2}=-1\left(x-\frac{3}{2}\right)\Rightarrow x+y=3$$

(c) The tangent line is horizontal at the points where $m=\frac{dy}{dx}=0$

$$\frac{y-x^2}{y^2-x}=0 \Rightarrow \quad y=x^2$$

Substituting this expression for y in the equation $x^3+y^3=3xy$

$$x^3+(x^2)^3=3x(x^2)$$

$$x^6-2x^3=0$$

$$x^3(x^3-2)=0$$

$$x=0 \text{ and } x=\sqrt[3]{2} \quad \text{the} \quad \text{first quadrant}$$

At $x=0 \Rightarrow y=0$ the point $(0,0)$

At $x=\sqrt[3]{2} \Rightarrow y=2^{2/3}$ the point $(2^{1/3}, 2^{2/3})$

Homework

Q1/Find dy/dx the following functions

1. $y=x^5-3x^3+1$
2. $y=\frac{5}{6}x^6-9x^4$
3. $y=\frac{x^{10}}{2}+\frac{x^5}{5}+6$
4. $f(x)=\frac{3}{x^2}+\frac{4}{x}$
5. $f(x)=\frac{1}{3x^3}-\frac{1}{2x^2}+1$
6. $f(x)=\frac{1}{x}-\frac{3}{x^2}$
7. $y=3x^{-2}-7x^{-1}$
8. $f(x)=\frac{2}{5x}-\frac{\sqrt{2}}{3x^2}$
9. $f(x)=\sqrt{3}(x^3-x)$
10. $y=(x^2+1)(2x^3+5)$
11. $g(x)=(x^2+3x)(x^3-9x)$
12. $f(x)=(6x^2+7)^2$
13. $f(x)=\left(\frac{1}{x}+3\right)\left(\frac{2}{x}+7\right)$
14. $f(x)=(x^3-8)\left(\frac{2}{x}-1\right)$
15. $f(x)=\frac{2x+7}{3x-1}$
16. $f(x)=\frac{3x^2}{x-2}$
17. $f(x)=\left(\frac{1}{x^2}+3\right)\left(\frac{2}{x^3}+x\right)$
18. $f(x)=\frac{2x^2+x+1}{x^2-3x+2}$
19. $f(x)=\frac{3x^2+7}{x^2-1}$
20. $g(x)=\left(x^2-\frac{1}{x}\right)\left(x-\frac{1}{x^2}\right)$
21. $y=\frac{x^2-17}{x^2+17}$
22. $4xy^2+3x^2y=2$
23. $xy^3+2y^3=x^2-4y^2$
24. $x^2-\sqrt{xy}-y=0$
25. $x^2y-xy^2+x^2=7$
26. $x^{-2/3}+y^{2/3}=1$
27. $x^4y+\sqrt{xy}=3$
28. $\sqrt{x}+\sqrt{y}=9$
29. $\sqrt{x+y}+\sqrt{x-y}=6$
30. $x\sqrt{y}+y\sqrt{x}=10$
31. $\sqrt[3]{xy}+3y=5\sqrt[3]{x}$
32. $x\sqrt{1+y}+y\sqrt{1+x}=4$
33. $\frac{x}{y}+\frac{y}{x}=5$
34. $\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}=6$
35. $\sqrt[3]{y}+\sqrt[4]{y}+\sqrt[5]{y}=4x$
36. $x^{1/n}+y^{1/n}=1$

Q 2/Derivative the following functions

① $y=(x^2+2)(x-4)(2x^2-5)$ ② $y=\left(\frac{1}{x^2}+1\right)(3x-1)(x^2-5x)$

Q 3/Find the equation of tangent and normal line for the following:

1. $f(x)=2x^2-7$, $(2,1)$ Sol. $y=8x-15$, $x+8y-10=0$
2. $f(x)=5+2x-x^2$, $(0,5)$ Sol. $y=2x+5$, $x+2y=10$
3. $f(x)=x^2+x+1$, $(1,3)$ Sol. $y=3x$, $x+3y-10=0$
4. $x^2+xy+2y^2=28$, $(2,3)$ Sol. $y=\frac{-1}{2}x+4$, $y=2x-1$
5. $f(x)=\sqrt[3]{x}$, $(8,2)$ Sol. $x-12y+16=0$, $12x+y=98$
6. $f(x)=\frac{3}{2}\sqrt{4-x^2}$, $(0,3)$ Sol. $y=0$, $x=0$
7. $y=x^3-8x^2+9x+20$, $(4,-8)$ Sol. $y=-7x+20$, $x-7y=60$
8. $f(x)=\frac{x^2-1}{x^2+1}$, $(1,0)$ Sol. $y=x-1$, $y=1-x$
9. $\sqrt{2x}+\sqrt{3y}=5$, $(2,3)$ Sol. $y=-x+5$, $y=x+1$

Q 4/Find the points where the tangent line to the graph of $f(x)=2\sqrt{x}$ and $(1,2)$ crosses (a) the x-axis and (b) the y-axis

Sol. $(-1,0)$,$(0,1)$

Q 5/ Find the points where the normal line to the graph of $f(x)=\frac{2}{x}$ and $(1,2)$ crosses (a) the x-axis and (b) the y-axis

Sol. $(-3,0)$,$(0,3/2)$

Q 6/ At what point on the curve $y=x^2+8$ is the slope of tangent line 16? Then write the equation for this tangent line?

Sol. $(8,72), 16x-y-80=0$

Q 7/ Find a point where the tangent line to the graph of $f(x)=x-x^2$ is parallel to the line $x+y-2=0$. Then write the equation for this tangent line?

Sol. $(1,0)$, $y=1-x$

Q 8/ Find all points on the graph of $f(x)=(x-3)(x-2)$ at which the tangent line is horizontal.

Sol. $(\frac{5}{2},\frac{-1}{4})$

Q 9/ Find all points on the graph of $f(x)=x+x^{-1}$ at which the tangent line is horizontal.

Sol. $(1,2)$

Q 10/ Find all points on the graph of $f(x)=x^3-6x^2+9x+4$ at which the tangent line is horizontal.

Sol. $(1,8)$, $(3,4)$

3.6 Velocity and Acceleration

Derivatives can be related very easily to physics applications. We can relate it to the position function, usually denoted as s(t) , the velocity function denoted v(t), and the acceleration function denoted a(t). Notice that are a function of time Make sure you understand the difference between average and instantaneous. The average velocity can be described as the change between two points, thus giving you the slope of the line connecting these two points! While the instantaneous velocity gives you the slope at a single point in time, thus giving you the slope of the tangent line.

Average formulas

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$\text{Average acceleration} = \frac{\Delta v}{\Delta t} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Where "s" is the position at any time "t"

Instantaneous Formulas

$v(t) = s'(t)$ $\qquad$ $a(t) = v'(t) = s''(t)$

Where $v(t)$ is the first derivative of the position function and $a(t)$ is the first derivative of the velocity function. Also note it is the second derivative of the position function.

Notes

v(t) =(0) means the object is not moving.

v(t) = (+) means the object would be moving forward (positive)

v(t) = (−)means the object would be moving backward (negative)

a(t) = (0) means there is no change in the velocity

a(t) = (+) means the object is going faster (positive)

a(t) = (−) means the object is slowing down.

Example 1: If $s(t) = t^2 - 20$. Find the average velocity from t = 3 to t = 5.

Solution:

$$Average\ velocity = \frac{\Delta s}{\Delta t} = \frac{s(5) - s(3)}{5 - 3} = \frac{5 - (-11)}{2} = \frac{16}{2} = 8$$

Example 2: Find the velocity function and the acceleration at t = 2 for the function $s(t) = 2t^3 + 5t - 7$.

Solution:

$v(t) = s'(t) = 6t^2 + 5$

$v(2) = 6(4) + 5 = 29$

$a(t) = v'(t) = 12t$

$a(2) = 12(2) = 24$

Example 3: If a ball is thrown vertically upward with an initial velocity of 128 ft/sec, the ball's height after t seconds is $s(t) = 128t - 16t^2$.

a) What is the velocity function?

Solution: $v(t) = 128 - 32t$

b) What is the velocity when t = 2, 4, 6

Solution: $v(2) = 128 - 32(2) = 64$

$v(4) = 128 - 32(4) = 32$

$v(6) = 128 - 32(6) = -64$

c) At what time is the velocity $48\ ft/sec$? $16\ ft/sec$? $-48\ ft/sec$?

Solution: Set the velocity function to each of the above values!

$$48 = 128 - 32t \qquad 16 = 128 - 32t \qquad -48 = 128 - 32t$$

$$-80 = -32t \qquad -112 = -32t \qquad -176 = -32t$$

$$2.5 = t \qquad 3.5 = t \qquad 5.5 = t$$

d) When is the velocity zero?

Solution: Set the velocity function to zero and solve

$$0 = 128 - 32t$$

$$-128 = -32t$$

$$4 = t$$

e) What is the height of the ball at the time the velocity is 0?

Solution: let t = 4 in the position function

$s(4) = 128(4) - 16(16) = 512 - 256 = 256$

f) When does the ball hit the ground?

Solution: Set the position function to zero and solve!

$128t - 16t2 = 0$

$16t(8 - t) = 0$

At t = 0, or t = 8. At t = 0, you haven't thrown the ball yet! The ball will hit the ground in 8 seconds. You could have reasoned that it took 4 seconds to reach maximum height and another 4 seconds to come back down.

g) What is the velocity when the ball hits the ground?

Solution: Find $v(8)$ $v(8) = 128 - 32(8) = -128\ ft/sec$. It is negative because the ball is coming back down! Notice it has the same velocity as when it was thrown upward!

h) What is the acceleration function?

Solution: Find $a(t)$ by finding the derivative of $v(t)$.

$a(t) = -32\ ft/sec^2$

j) What is the acceleration at t = 3, 5, 8?

Solution: $a(3) = -32$ $a(5) = -32$ $a(8) = -32$

No matter what time it will always be -32. (This is the effect of gravity.)

Homework.

*q*❶/The position of a particle on a line is given by $s(t) = t^3 - 3t^2 - 6t + 5$, where t is measured in seconds and s is measured in feet. Find

a. The velocity of the particle at the end of 2 seconds. *Sol.* −6 ft/sec

b. The acceleration of the particle at the end of 2 seconds. *Sol.* 6 ft/sec^2

*q*❷/ The formula $s(t) = -4.9t^2 + 49t + 15$ gives the height in meters of an object after it is thrown vertically upward from a point 15 meters above the ground at a velocity of 49 m/sec. How high above the ground will the object reach? *Sol.* 137.5 m

*q*❸/ A ball is hit straight upward with an initial velocity of 256 feet per second. the ball's height after time t seconds is $s(t) = 256t - 16t^2$.

a) What is the velocity function? *Sol.* $256 - 32t$

b) What is the velocity at t = 6, t = 8, t = 10? *Sol.* −64

c) What is the acceleration function? *Sol.* −32

d) At what time does the ball hit the ground? *Sol.* 16

e) What is the velocity of the ball when it hits the ground? *Sol.* −256

f) What time does it reach maximum height? *Sol.* 8

g) What is the maximum height? *Sol.* 1024

*q*❹/ Find the velocity and acceleration at t = 0,1,2 for the following :

$a)\ s = \dfrac{1}{t+1}$ *Sol.* -1,-1/4 , -1/9 , 2,1/4,2/27

$b)\ s = t^2 + 2t + 5$ *Sol.* 2,4 ,6 , 2,2,2

$c)\ s = t^2(t-1)$ *Sol.* 0,1,8 ,−2,4,0

q❺/ A point moves in a straight line so that its distance s (in meters) after time t (in seconds) is $s = 4t^2 - 16t + 12$, Find

1: the average velocity in the interval [1,7] Sol. 16

2: the velocity at t=3 Sol. 8

q❻/ The position of a body (in feet) at time of t seconds is $s = t^3 - 6t^2 + 9t$ Find the body's acceleration each time its velocity is zero.

Sol. $-6\ ft/sec^2$, $6\ ft/sec^2$

3-7 Chain rule

If y is a differentiable function of u, $y = f(u)$ and u is differentiable function of x, $u = f(x)$ then y is a differentiable function of x :

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

If y is a differentiable function of u, $\mathrm{y} = \mathrm{f(u)}$ and x is differentiable function of u, $\mathrm{x} = \mathrm{f(u)}$ then y is a differentiable function of x :

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \div \frac{dx}{du}}$$

Example 1 : If $y = u^3$ and $u = x^2 + 5x$ find $\frac{dy}{dx}$

Solution : $\frac{dy}{du} = 2u^2$, $\frac{du}{dx} = 2x + 5$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u^2)(2x+5) = 2(x^2+5x)^2(2x+5)$$

Example 2 : If $y = t^3$ and $x = 2t^2 + 3t - 1$ find $\frac{dy}{dx}$

Solution : $\frac{dy}{dt} = 2t^2$, $\frac{dx}{dt} = 2t + 3$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = (2t^2) \cdot \frac{1}{2t+3} = \frac{2t^2}{2t+3}$$

Example 3 : find $\frac{dy}{dx}$ at $x = 0$ for $y = \frac{1}{1+u}$ and $u = (3x+1)^3$

Solution : $\frac{dy}{dt} = \frac{(1+u)(0)-(1)(1)}{(1+u)^2} = \frac{-1}{(1+u)^2}$

$\frac{du}{dx} = 3(3x+1)^2(3) = 9(3x+1)^2$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{-1}{(1+u)^2} \cdot 9(3x+1)^2$

$at\ x = 0 \quad \Rightarrow u = (0+1)^3 \Rightarrow u = 1$

$\frac{dy}{dx} = \frac{-9(0+1)^2}{(1+1)^2} = \frac{-9}{4}$

Note: If y is a differentiable function of v, $y = f(v)$ and v is differentiable function of u, $v = f(u)$ and u is differentiable function of x, $u = f(x)$ then y is a differentiable function of x :

$$\boxed{\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}}$$

Example 1 : If $y = v^3, v = \sqrt{u}\ and\ u = 1 + x^2$ find $\frac{dy}{dx}$

Solution : $\frac{dy}{dv} = 3v^2 \quad , \frac{dv}{du} = \frac{1}{2\sqrt{u}} \quad , \frac{dv}{du} = 2x$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = (2v^2) \cdot \left(\frac{1}{2\sqrt{u}}\right) \cdot (2x)$$

$$= \left(2\left(\sqrt{u}\right)^2\right) \cdot \left(\frac{1}{2\sqrt{u}}\right) \cdot (2x)$$

$$= \left(3(1+x^2)\right) \cdot \left(\frac{1}{2\sqrt{1+x^2}}\right) \cdot (2x) = 3x\sqrt{1+x^2}$$

Homework

q❶ Using chain rule to find $\frac{dy}{dx}$

① $y = \sqrt{u}\ ,u = x^2 + x + 1$ Sol. $\frac{dy}{dx} = \frac{2x+1}{2\sqrt{x^2+x+1}}$

② $y = u^3 - u^{1/2}\ ,u = x^2 + 2x$ Sol. $\frac{dy}{dx} = (6x-6)(x^2+2x) - \frac{2x+2}{\sqrt{x^2+2x}}$

③ $y = u^{-5}\ ,u = x^4 + 1$ Sol. $\frac{dy}{dx} = \frac{-20x^3}{(x^4+1)^6}$

④ $y = (t^2+2)^2\ ,t = x^{1/2}$ Sol. $\frac{dy}{dx} = 2(x+2)$

⑤ $y = \sin(3x+1)\ ,u = 3x + 1$ Sol. $\frac{dy}{dx} = 3\cos(3x+1)$

⑥ $y = \cos\sqrt{3}\,x\ ,u = \sqrt{3x}$ Sol. $\frac{dy}{dx} = -\sqrt{3}\sin\sqrt{3}\,x$

⑦ $y = \left(\frac{\sin x}{1+\cos x}\right)^2\ ,u = \frac{sin\,x}{1+cos\,x}$ Sol. $\frac{dy}{dx} = \frac{2\,sin\,x}{(1+cos\,x)^2}$

⑧ $y = \cos(\sin x)\ ,u = sin\,x$ Sol. $\frac{dy}{dx} = -\sin(\sin x)\cos x$

q❷ find $\frac{dy}{dx}$ at $t = 1$ for $y = u^3 - u^2\ ,u = \frac{1-x}{1+x}$ *and* $x = 2t - 5$

Sol. $\frac{dy}{dx} = -16$